Hooke, orbital motion, and Newton's Principia

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(Received 23 July 1992; accepted 30 August 1993)

A detailed analysis is given of a 1685 graphical construction by Robert Hooke for the polygonal path of a body moving in a periodically pulsed radial field of force. In this example the force varies linearly with the distance from the center. Hooke's method is based directly on his original idea from the mid-1660s that the orbital motion of a planet is determined by *compounding* its tangential velocity with a radial velocity *impressed* by the gravitational attraction of the sun at the center. This hypothesis corresponds to the second law of motion, as formulated two decades later by Newton, and its *geometrical* implementation constitutes the cornerstone of Newton's *Principia*. Hooke's diagram represents the first known accurate graphical evaluation of an orbit in a central field of force, and it gives evidence that he demonstrated that his resulting discrete orbit is an approximate ellipse centered at the origin of the field of force. A comparable calculation to obtain orbits for an inverse square force, which Hooke had conjectured to be the gravitational force, has not been found among his unpublished papers. Such a calculation is carried out here numerically with the Newton–Hooke geometrical construction. It is shown that for orbits of comparable or larger eccentricity than Hooke's example, a graphical approach runs into convergence difficulties due to the singularity of the gravitational force at the origin. This may help resolve the long-standing mystery why Hooke never published his controversial claim that he had demonstrated that an attractive force, which is "···in a duplicate proportion to the Distance from the Center Reciprocall···" implies elliptic orbits.

I. INTRODUCTION

One of the most fascinating questions in the history of science is the role that Robert Hooke played in the development of dynamics and the theory of gravitation during the 17th century, which culminated with Newton's masterpiece the *Principia*.[1] Hooke, one of the most prolific and inventive scientists of all times, made fundamental contributions to physics, astronomy, chemistry, geology, biology, and meteorology;[2] one-third of the 15 volumes of Gunther's *Early Science in Oxford*[3] are dedicated to his work.[4,5] However, in spite of the profound influence which Hooke had, particularly on Newton's work, shortly after his death in 1703 he was nearly completely forgotten until about the turn of this century. In an influential book entitled *The Science of Mechanics, a Critical and Historical Account of its Development*, first published in 1883, Ernst Mach devoted only a few lines to Hooke, although he perceptively stated that "···Hooke really approached nearest to Newton's conception, though he never completely reached the latter's altitude of view" (Ref. 6). Ten years later, the publication of some of the correspondence between Hooke and Newton by Rouse Ball,[7] and the subsequent discovery of two additional letters published by Pelseneer[8] and Koyré,[9] initiated a reappraisal of Hooke's contributions to mechanics[10–16] which continues to the present time,[5] although a general consensus about Hooke's importance has not been reached. In contrast to Mach,[6] Dugas recognized Hooke's crucial role in his book *Mechanics in the Seventeenth Century*,[17] although some recent accounts of the development of mechanics still ignore Hooke's role completely.[18,19] Meanwhile, most physicists and mathematicians have remained unaware of these developments, as can be seen by reading textbooks or journals that cover classical mechanics, where Hooke is mentioned only in relation with the law of elasticity.[20] A notable exception is a recent book by Arnol'd entitled *Huygens and Barrow, Newton and Hooke*.[21]

Hooke's profound physical intuition, which was guided by his numerous experiments, thought to be on the order of several hundred,[22] led him during the middle 1660s to a correct *qualitative* formulation of the principles of dynamics as applied to celestial mechanics. Hooke stated that the orbital motion of a planet is determined by *compounding* its tangential velocity with the radial velocity impressed by the gravitational attraction of the sun. Further, he formulated the concept of universal gravitational attraction at a distance and deduced the inverse square law, based on the conjecture that the origin of gravity was due to periodic pulses from matter, by analogy with the emission of light and sound.[23,24] While crediting Hooke with some of these ideas, and acknowledging his influence on Newton, historians of science, with few exceptions,[10] have generally concluded that he was unable to obtain a *quantitative* or *mathematical* formulation of his principles. This judgment echoes the charge made already by Newton in a June 20, 1686 letter to Haley that stated:

> "···Borell did something in it and wrote modestly, he [Hooke] has done nothing and yet written in such a way as if he knew and had sufficiently hinted all but what remained to be determined by the drudgery of calculations and observations, excusing himself from that labour by reason of his other business: whereas he should rather have excused himself by reason of his inability. For it is plain by his words he knew not how to go about it. Now is not this very fine? Mathematicians that find out, settle and do all the business must content themselves with being nothing but dry calculators and drudges and another that does nothing but pretend and grasp at all things must carry

away all the invention as well of those that were to follow him as of those that went before···For as Borell wrote long before him that by a tendency of the Planets towards the sun like that of gravity or magnetism the Planets would move in Ellipses, so Bullialdus wrote that all force respecting the sun as its center and depending on matter must be reciprocally in duplicate ratio of the distance from the center···" (Ref. 25).

In his letter, Newton was focusing his fury about accusations from Hooke that he had plagiarized the discovery of the inverse square radial dependence of the gravitational force. In his capacity as editor of the *Principia*, Halley had tactfully written

"···that Mr Hook has some pretensions upon the invention of the rule of the decrease of Gravity, being reciprocally as the squares of the distances from the Center. He sais you had the notion from him, though he owns the Demonstrations of the Curves generated thereby to be wholy your own···"

Newton also criticized Hooke's claim that this radial dependence of the force of gravity implied elliptic orbits for the planets moving around the sun, pointing out that Hooke had concluded incorrectly that the velocity varied inversely with the distance. What Newton conveniently forgot in his lengthy diatribe to discredit Hooke, is that in their 1679/80 correspondence, Hooke had described *correctly* some of the principles of orbital motion that led Newton to the discovery of Kepler's area law,[14,15] and to a deeper understanding of orbital dynamics.[26] Later, Newton referred to Hooke in his *System of the World*, the less mathematical treatment of Book III of the *Principia*, lumping him together with other well-known philosophers whose speculations about the motion of the planets were *wrong*:

"The later philosophers *pretend* to account for it either by the action of certain vortices, as *Kepler* and *Descartes*; or by some other principle of impulse or attraction, as *Borelli, Hooke*, and others of our nation···"

Actually, Hooke arrived at his remarkable physical insights in dynamics by careful observation of *mechanical analogs* of celestial motion, and not by just *guessing*. With the notable exception of Lohne[11] and Arnol'd,[21] this misconception that Hooke's discoveries of the principles of mechanics and the law of gravitational force were pure *guesses*, or based somehow on incorrect mathematical reasoning, has been repeated in many accounts by Newtonian scholars. To quote one typical example, Hall[16] states:

"One sees his (Newton's) point: Hooke had been almost as vague as Borelli, and certainly could never have produced dynamical demonstrations applicable to planetary motion: yet we may allow that the idea of a terrestrial projectile becoming a satellite in elliptical motion was Hooke's own, though a 'guess' indeed as Newton rightly called it···"

The sentiment by many historians of science that Hooke did not have any sound basis for his physical principles is often expressed by quoting the 18th century French mathematician Alexis-Claude Clairaut, who, although considered a supporter of Hooke, stated that

"Hooke's examples serve to show what a distance there is between a truth that is glimpsed and a truth that is demonstrated···" (Refs. 7, 14, 15).

However, from the outset, Hooke's dynamical principles were grounded on careful experiments and observations of well-designed mechanical systems that could serve as analogs of celestial dynamics. The best documented example is the circular or conical pendulum,[27] but there is evidence that he also studied the dynamics of balls rolling on various surfaces of revolution. These serve as approximate *analog models* for different central attractive forces. Indeed, for a long time, Hooke had been applying the maxim,

"···that Nature seems to take similar Ways for producing similar Effects; without granting of which we cannot reason or make any Conclusion from similar Operations." (Ref. 23).

However, he cautions that[28]

"*Omne simile non est idem.*" (Everything that looks the same is not the same) (Ref. 23).

Recently, in an article entitled *Robert Hooke and the Dynamics of Motion in a Curved Path*, Pugliese[29] reproduced a remarkable diagram shown in Fig. 1 which is kept among the unpublished manuscripts of Hooke in the Wren Library at Trinity College, Cambridge.[30] This diagram, which is part of an unfinished document entitled *The Laws of Circular Motion*,[31] shows a graphical construction of a segment of an orbit for a body moving in a central field of force. On a page of the manuscript associated with this diagram a date is inscribed: Sept. 1, 1685. It turns out that this date is important in relating this work to Newton's earliest draft of the *Principia*, the manuscript *De Motu Corporum Gyrum*,[32] which was registered by Edmond Halley at the Royal Society in November 1684. In Hooke's example of orbital dynamics, the force varies linearly with the distance from the center, and in the handwritten text associated with the diagram (see also the Appendix), he states that the corresponding orbit is an *ellipse*. In his article, Pugliese analyzes Hooke's graphical construction only for the special case of circular motion, and then implies that Hooke somehow failed in his attempt to generalize the construction to noncircular motion, as shown in the diagram, Fig. 1, concluding that "···Hooke claims, but certainly does not demonstrate [the path] to be elliptical···" Earlier in the paper, Pugliese asserted that "···there can be no doubt··· that [Hooke] could not have taken his dynamical principle so far as Newton." He ends with the comment that Hooke "···does not seem to have ever come to a full appreciation of the magnitude of the step from his ideas to Newton's achievements···" These comments and conclusions appear to reinforce the conventional wisdom among historians of science about Hooke's mathematical limitations. However, a careful analysis of Hooke's diagram, Fig. 1, leads to quite *opposite* conclusions.

In this paper I will show that Hooke's graphical solution, Fig. 1, for the orbit of a body in a radial field of force that varies linearly with the distance from the center, is based on precisely the same *geometrical construction* developed by Newton in Theorem 1 of *De Motu*,[32] which became later Proposition I, Theorem I, Book I of the *Principia*. Further, this geometrical construction is effectively the *mathematical formulation* of the principles of dynamics which Hooke had been proposing during the past 20 yr. Contrary to Pugliese's assertion, I will show that in his diagram, Fig. 1, Hooke gives at least three graphical dem-

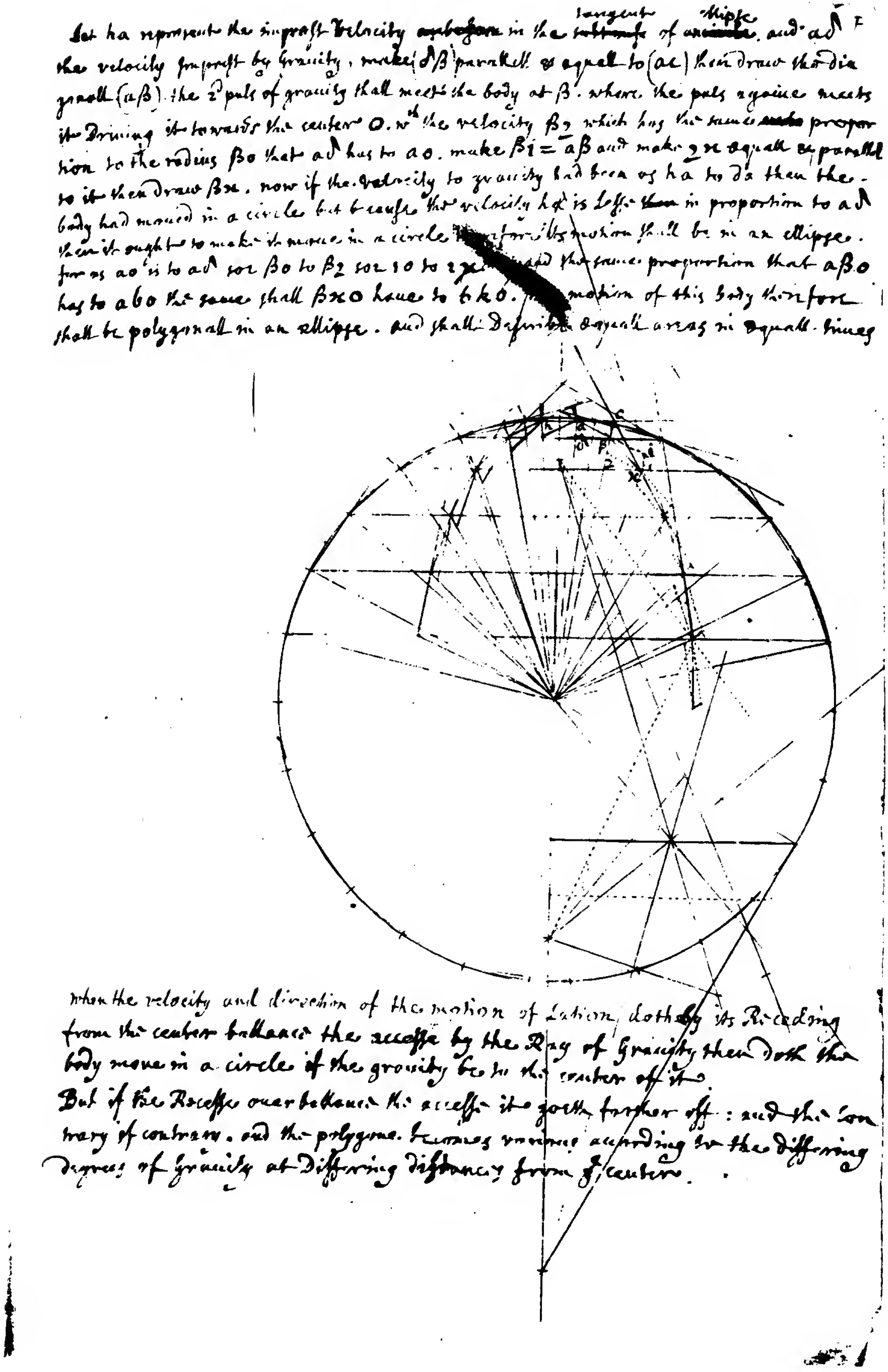

Fig. 1. Hooke's September 1685 geometrical construction and graphical evaluation of orbital motion in a central field of force which varies linearly with distance. The original copy is in the collection of Hooke's manuscripts in the Trinity College Library, Cambridge, MS O.11a.1/16. Reproduced by permission of the Master and Fellows of Trinity College.

onstrations that the vertices of the resulting polygonal orbit lie on an *ellipse*. Moreover, this property is exact, as can be shown by an affine transformation.[33]

While the extent to which Hooke and other members of the Royal Society were aware of Newton's manuscript[34] is uncertain, there are at least two letters which indicate that Hooke may have seen the copy of the *De Motu* shortly after it had been registered at the Royal Society in November 1684. A month later Flamsteed wrote to Newton[25] that

> "··· I am obliged by your kind concession of ye perusall of your papers, tho I believe I shall not get a sight of them till our common friend Mr. Hooke and the rest of the towne have been first satisfied···"

Later, on June 29, 1686, Halley wrote to Newton[25] that

> "···it [*de Motu*] has been entered upon the Register books of the Society as all this past Mr. Hook was acquainted with it; ···"

If Hooke had indeed seen the 1684 version of *De Motu*,[32] he would have recognized that Newton had implemented geometrically his dynamical principle of *compounding* a tangential velocity with an impressed radial *velocity* due to a center of attraction. On the first page of this manuscript, Newton draws a diagram and describes a geometrical construction (similar to Fig. 3) that embodies Hooke's dynamical principles. He then applies it to prove Theorem 1, that "all bodies circulating a centre [of force] sweep out areas proportional to the times." Regardless of whether Hooke had seen *De Motu*, his own description of the geometrical construction described in the handwritten text in Fig. 1 (see also the Appendix and Fig. 3) is given directly in the physical terms that he had used for the past 20 yr:

> "···let $h\alpha$ represent the impressed velocity in the tangent of an ellipse and $\alpha\delta$ the velocity impressed by Gravity···the puls of Gravity [Hooke's ad hoc theory of periodic gravitational pulses] driving it toward the center···then draw the diagonal β [*compounding* the two *vector* velocities shown as displacements in the diagram]···"

This is not a mere translation of Newton's mathematical description in Latin of the corresponding geometric construction associated with Theorem 1 of *De Motu* (for comparison see Herivel's translation)[35] which later became the cornerstone of the *Principia*, as Proposition 1, Theorem 1 of Book 1. Further, Hooke then proceeds in quite a different way from Newton, applying the geometrical construction in a *novel* graphical manner to obtain an orbital path in a central field of force that varies linearly with distance.

The orbital problem posed by Hooke is to add two vectors, a tangential velocity, and a velocity change in the radial direction impressed by an attractive central force, both of which are varying continuously in time along the orbit. Previously, Galileo had solved the corresponding problem for the case that the change of velocity due to gravity is along a fixed (vertical) direction in space.[36] In this special case the vertical and horizontal components of motion are *separable*, but this is not the case for central force. The new mathematical idea developed by Newton and by Hooke was to assume that the force consisted of *pulses* applied at periodic intervals of time. While Newton considered a finite time interval between impulses to be only an approximation, and took the limit as this interval becomes vanishingly small, Hooke had conjectured earlier that the gravitational force consisted of just such *periodic pulses*.[37] It is from this "supposition" that Hooke deduced that the gravitational force decreased as the inverse of the square of the distance from the center, in analogy with similar phenomena like pulsed radiation of light and sound.[23,24] This crucial idea of replacing a continuous force by a series of short impulses varying periodically in time, in order to treat accelerated motion mathematically, can be traced back to the work of a Dutch Latin schoolteacher, Isaac Beeckman.[38] In collaboration with Descartes in 1618, he succeeded in this manner to calculate accelerated motion of a body in a constant field force. Surprisingly, this fruitful mathematical method was not generalized to central forces until about 66 yr later by Newton in *De Motu*, and by Hooke, who applied it graphically to obtain an accurate *numerical* calculation of a dynamical orbit in a central field of force. In his diagram Hooke also points out the essential feature that this geometrical construction leads to Kepler's area law, although its dynamical origin due to the restriction of forces which are central was demonstrated only by Newton in the *De Motu*, and later in the *Principia*.

Unfortunately, a comparable graphical calculation by Hooke of the orbital path for the force of gravitational attraction has not been found. This is particularly surprising, because such a calculation for the $1/r^2$ dependent force is almost as easy to do by Hooke's method as for a force depending linearly on r. Further, from the onset Hooke's main goal had been precisely to *calculate* planetary orbits "···to the greatest exactness and certainty that can be desired···" (Ref. 27). Indeed in 1684, Hooke had claimed to Edmond Halley and Christopher Wren that he had succeeded in demonstrating that an inverse square law of force implied elliptic orbits.[39] However, according to his early biographer Richard Waller,[22] many important papers of Hooke have been lost. It is straightforward to reproduce the inverse square calculation with Hooke's graphical method; the results are discussed below. In particular, I will show that when the trajectory approaches the center of force too closely, as is the case for orbits of high eccentricity like the comets, Hooke's graphical application runs into a problem of convergence. It is likely that this difficulty may have been encountered by Hooke, which would account for his reluctance to publish his results before he had a chance to resolve the difficulty. The appearance of the *Principia* shortly thereafter with a solution to his problem—taking the limit of vanishingly small time steps—must have been a stunning but perhaps not unexpected blow to Hooke. As pointed out by Lohne, after the fall of 1686 "···we can see him feverishly active to assert his claims of priority." (Ref. 11).

In order to understand Hooke's contributions to the genesis of Newton's *Principia*, and the origin of Hooke's diagram, Fig. 1, it is important to review certain parts of the 1679/80 correspondence of Hooke and Newton, and the 1686 correspondence between Halley and Newton. Aspects of this correspondence acquire new significance in light of Hooke's diagram, which I will endeavor to explain in Sec. II. In Sec. III a detailed discussion is given of Hooke's geometrical construction and its graphical application to the case of a radial force. An algebraic formulation of this construction is given in Sec. IV, which leads to *discrete* equations of motion for a body acted on by impulsive central forces. In the limit of small time steps, algebraic proofs are given for Newton's theorems in the *Principia* that elliptic motion implies linear or inverse square radial depen-

dence of the central force. Section V describes the numerical application of these equations to the inverse square law, for initial conditions similar to those that Hooke took for the linear force law. A summary and conclusions are presented in Sec. VI. The Appendix gives an annotated version of the handwritten text describing Hooke's 1685 diagram, Fig. 1.

II. BACKGROUND

On May 23, 1666, Hooke gave a remarkable lecture to the members of the newly founded Royal Society entitled *Planetary Movements as a Mechanical Problem*,[27] in which he proposed that the Keplerian elliptic orbits of planets around the sun could be *calculated*[40] by *compounding* an inertial straight line motion with an *inflection* toward the center due to an attractive property of the sun. This is the first time that some of the essential dynamical principles of planetary motion were publicly and unambiguously stated, nearly two decades before they were implemented in mathematical form by Newton in his celebrated *Principia*. Hooke's paper, registered by the Royal Society, starts with the statement that

> "I have often wondered, why the planets should move about the sun according to Copernicus's supposition, being not included in any solid orbs (which the ancients possibly for this reason might embrace) nor tied to it, as their centre, by any visible strings; and neither depart from it beyond such a degree, nor yet move in a straight line, as all bodies, that have one single impulse, ought to do···"

Hooke then dismisses a theory, partly due to Borell,[41] that the impressed force is due to a medium of variable density acting on the planetary body, and states his own idea that

> "···the second case of inflecting a direct motion into a curve may be from an attractive body placed in the centre; whereby it continually endeavours to attract or draw it to itself. For if such a *principle* be supposed, all the phenomena of the planets seem possible to be explained by the common principle of mechanic motions; and possibly the prosecuting of this speculation may give us a true hypothesis of their motion, and from some few observations, their motions may be so far brought to a certainty, that we may be able to calculate them to the greatest exactness and certainty that can be desired··· This inflexion of a direct motion into a curve by a supervening attractive principle I shall endeavour to explicate from some experiments with a pendulous body: not that I suppose the attraction of the sun to be exactly according to the same degrees as they are in the pendulum···"

Hooke had in mind that a conical or circular pendulum could serve as a *mechanical analog* to demonstrate the principles of orbital motion by projecting its motion on a plane perpendicular to the suspension.[42] This analog model generalizes the demonstration of circular motion by Descartes[43] of a stone revolving on a sling.[44] Hooke discussed the theory of the circular pendulum pointing out that the effective radial force

> "··· is greater and greater, according as it is farther and farther removed from the center, which seems to be otherwise in the attraction of the sun···But however it be, the compounding this motion with a direct or straight motion just crossing it, may serve to explicate this hypothesis, though all the appearances of it are not exactly the same···"

During this time, Hooke was the Curator of the newly founded Royal Society, and one of his main tasks was to present weekly scientific experiments. He gave a demonstration to the members with

> "··· a pendulum fastened to the roof of the room with a large wooden ball of lignum vitae on the end of it ···"

This is an impressive demonstration of nearly closed elliptic orbits, projected on a plane normal to the axis of suspension. Hooke carefully observed that

> "··· the progression of the auges [apsides] are very evident···"

As Hooke demonstrated mathematically, the horizontal force in this case increases approximately linearly with the distance from the axis.[45] This axis coincides with the center of the ellipse, while for planetary motion the center of force is located at a focal point of the ellipse. Further the period of the pendulum swing is nearly independent of the size of the orbit, as he discovered later for the oscillation of springs.[20] This is in striking contrast with the dependence of the period of the planets on the distance of the sun, described quantitatively by Kepler's third law, that the square of the period varies with the cube of the major axis of the ellipse. Hooke also added a smaller pendulum attached to the ball to demonstrate, although less successfully, the motion of a planet like the moon around the earth. What is particularly interesting in this demonstration is that it shows Hooke's understanding of the universal character of the gravitational force which he explicitly enunciated later on. He had proposed and carried out several experiments, which were inconclusive, to determine experimentally how the gravitational force varies with distance both above and below the surface of the earth.[46]

Hooke continued to present further developments of his ideas in his Cutlerian lectures delivered at the regular meetings of the Royal Society. The first of these lectures, in 1670, was on *An Attempt to prove the motion of the Earth by Observations* (Ref. 47) by trying to observe the parallax of stars due to the earth's orbit around the sun, to

> "···furnish the Learned with an *experimentum crucis* to determine between the Tychonick and the Copernican hypothesis···"

He had erected a telescope vertically in his own quarters at Gresham College,[48] and was ready to start his observations by October, 1668. At the end of his lecture, which was published later as a short tract[49] in 1674, he restated his *principles* of dynamics, and formulated clearly the principle of universal gravitational attraction. He stated that

> "···At which time also I shall explain a *System of the World* differing in many particulars from any yet known, answering in all things to the common Rules of Mechanical Motion: This depends on three Suppositions. First, That all Celestial Bodies whatsoever, have an attraction or gravitating power towards their own Center, whereby they attract not only their own part, and keep them from flying from them, as we may observe the Earth to do, but that they do also attract all the other Celestial Bodies that are within the sphere of their activity; and consequently that not only the Sun and Moon have an influence upon the body and motion of the Earth, and the Earth upon

them, but that Mercury also Venus, Mars, Saturn and Jupiter by their attractive powers have a considerable influence upon its motion as in the same manner the corresponding attractive power of the Earth hath a considerable influence upon every one of their motions also. The second supposition is this, That all bodies whatsoever that are put into a direct and simple motion, will so continue to move forward in a straigth line, till they are by some other effectual power deflected and bent into a Motion, describing a Circle, Ellipsis, or some other more compounded Curve Line. The third supposition is, That these attractive powers are so much the more powerful in operating by how much the neare the body wrought upon is to their own Centers. Now what these several degrees are I have not yet experimentally verified···"

Finally, he concluded his account with the prophetic remark that

"···"This I only hint at present to such as have ability and opportunity of prosecuting this Inquiry, and are not wanting of industry for observing and calculating, wishing heartily such may be found, having myself many other things in hand which I would first compleat and therefore cannot so well attend it. But this I durst promise the Undertaker, that he will find all the great Motions of the World to be influenced by this Principle, and that the true understanding thereof will be the true perfection of Astronomy···"

In light of Newton's unkind description of Hooke which is reproduced in Sec. I, it is worthwhile to remember here that Hooke was a poorly paid employee of the Royal Society, whose aristocratic members often "ordered" him around to do this or that demonstration practically every week of the year.[3] Hooke did not have the advantages of inherited wealth like his early mentor, Robert Boyle,[50] or a well paid academic chair like Newton's, which would have give him the leisure to follow his own intellectual pursuits.

Hooke's fifth Cutlerian lecture was entitled *Cometa*,[51] and it contained observations of the comet in April 1677 and those of 1664 and 1665 showing that he tried systematically by observation to determine the motion and properties of comets. Hooke had suggested earlier that comets might move in close orbits around the sun, and that therefore sightings of comets at different times might actually be due to one and the same comet. This was demonstrated more than two decades later by calculations of his younger friend and colleague Edmond Halley.[52] Samuel Pepys recalls in his *Diary* that on March 1, 1665 he went

"··· to Gresham College where Mr. Hooke read a second very curious lecture about the late Comet; among other things proving very probably that this is the very same Comet that appeared before in the year 1618, and that in such a time probably it will appear again, which is a very new opinion; but all will be in print···" (Ref. 53).

Hooke, like many contemporary astronomers, was particularly interested in observing comets, which he described in great detail, and he speculated at length about the constitution and orbital motion of these celestial bodies. The critical question was whether comets obeyed the same principles of motion and of gravitational attraction as the planets. Hooke wavered on this point, because he could not understand why the tail of a comet is always directed away from the sun, and he therefore suggested that additional *repulsive* forces were acting in this case.[54] It appears that up to this time, Hooke apparently did not commit himself to the radial dependence of the gravitational attraction, except to state that it should weaken with distance. However, he always stressed in his lectures that this dependence should be obtained from experiments, which he tried unsuccessfully to carry out.

Then in 1679/80, Hooke discussed his ideas with Newton in the celebrated correspondence[25] which was partially published at the turn of the century by Rouse Ball.[7] Hooke had become secretary of the Royal Society, and the avowed purpose in his first letter was to reestablish contact with Newton, which had become strained during their earlier controversy on optics, and to elicit some reaction to his current physical hypotheses. In particular, on November 24, 1679 he writes that

"···For my own part I shall take it as a great favour if you please to communicate by Letter your objections against any hypothesis or opinion of mine, And particularly if you will let me know your thoughts of that of compounding the celestiall motions of the planetts of a direct motion by the tangent and an attractive motion towards the centrall body···"

It appears from this and the later correspondence with Newton, that one of the principal reasons why Hooke initiated this correspondence was that he had been unable to make progress in expressing mathematically his physical principles of celestial mechanics, and that he wanted to get some help from Newton whose great mathematical abilities had by then become known to several members of the Royal Society. Hooke's diary shows that he had gone to other mathematicians for help without success.[55] The discussion in the correspondence about the motion of a body inside the earth was mostly relevant to him insofar as it clarified issues of orbital dynamics for central forces which he had been studying with mechanical analogs for the past 14 yrs. Therefore, when Newton sent him a letter on December 13, 1679 with a drawing of an orbit under the assumption of a constant radial force,[8] Hooke responded within a few days that

"···Your Calculation of the Curve by a body attracted by an equall power at all Distances from the center Such as that of a ball Rouling in an inverted Concave Cone is right and the two auges [apsides] will not unite for about a third of a Revolution···"

What a revelation Newton's letter must have been to Hooke. He demonstrates by his comment that he knew that the effective radial force in an inverted cone is (approximately) a constant, and he therefore realized immediately that Newton evidently had a method to calculate an orbit which he had observed previously only experimentally in one of his mechanical analogs for celestial motion.[56] Hooke continues

"···But my supposition is that the Attraction always is in a duplicate proportion to the Distance from the Center Reciprocall, and Consequently that the Velocity will be in a subduplicate proportion to the Attraction and Consequently as Kepler Supposes Reciprocall to the Distance···"

Here Hooke announced for the first time that he believed that the force of gravity dependent inversely on the square of the radial distance. However, he made an error quoting Kepler on the incorrect dependence of the velocity on distance.[31] Newton later pounced on this error, claiming that

Hooke did not understand anything about orbital motion due to an inverse square dependent radial force, ignoring the fact that in this letter Hooke was correct in his *supposition*

"···that with Such an attraction [inverse-square law force] the *auges* [apsides] will unite in the same part of the Circle and that the neerest point of accesse to the center will be opposite to the furthest Distant. Which I conceive doth very Intelligibly and truly make out all the Appearances of the Heavens···"

There is nowhere in this letter any evidence that Hooke knew at that time how to demonstrate his supposition *mathematically.* Instead, he goes on stating that

"(though in truth I agree with You that Explicating the Curve in which a body Descending to the Center of the Earth would circumgyrare were a Speculation of noe use Yet), the finding out the proprietys of Curve made by two such principles will be of great Concerne to Mankind··· This Curve truly calculated will shew the error of those many lame shifts [ad hoc approximations] made use of by astronomers to approach the true motion of the planets with their tables···"

Finally, in a letter on January 17, 1679/80 Hooke states again that

"···I doubt not but that by your excellent method you will easily find what that Curve must be, and its proprietys, and suggest a physicall Reason of this proportion···"

These remarks indicate that at the time of this correspondence Hooke did not know how to calculate the general orbital motion in a central field of force; his remarkable physical understanding and hypothesis were based on mechanical *analog* models. However, asking Newton for mathematical help turned out to have been his capital mistake. Newton solved his problem, but never acknowledged Hooke's semimal contributions or replied to his last letter. This has been pointed out not only by Hooke's defenders,[10,11] but also by several Newtonian scholors.[9,12,14,15] Nevertheless, many scholars believe that Newton was justified in denying credit to Hooke because he was only *guessing* instead of providing *mathematical proofs.* Hooke's often quoted error of claiming that the orbital velocity was inverse with the distance, "···as Kepler Supposes···" shows that in 1679 he had not yet understood Kepler's area law, as was also the case with most of his contemporaries.[55] Only a few historians of science seem to appreciate that Hooke's physical insights came primarily from observations and experiments with mechanical analog systems.[11]

On November 28, 1679 Newton had written to Hooke

"···that I did not before the receipt of your last letter [sent four days earlier], so much as heare (that I remember) of your Hypotheses of compounding the celestial motion of the Planets, of a direct motion by the tangent to the curve··· If I were not so unhappy to be unacquainted with your Hypotheses above mentioned···" (Ref. 25).

In the same letter Newton congratulated Hooke

"···that so considerable a discovery as you made of the earth's annual parallax is seconded by Mr. Flamstead's Observation···"

In his letter, Hooke had not told Newton anything about his own apparent discovery of the earth's parallax, but had commented only briefly that Flamstead

"···hath confirmed the parallax of the orb of the earth···"

It appears therefore that Newton was familiar with Hooke's tract on *An Attempt to prove the motion of Earth by Observations*,[47] as he later admitted in his correspondence with Halley. Indeed, Hooke was not convinced of Newton's denials that he was acquainted with his ideas, and he wrote later on about this affair,[57] that

"···Newton pretends he knew not Hooke's Hypoth. as by his Answer to the former, dated November 28. 1679···"

In focusing primarily on the priority conflict between Hooke and Newton concerning the discovery of the inverse square law of gravitational attraction, it has been often overlooked, as it conveniently was by Newton in his letters to Halley,[25] that at least since 1666 Hooke had stated correctly fundamental physical aspects of the first and second laws of motion as these laws apply to planetary orbits with a gravitational attraction to the sun. He had not appealed to Cartesian vortices or to an intermediary medium or ether, as was fashionable with his contemporaries who apparently could not accept the concept of an action at a distance, in spite of such familiar evidence as magnetism and terrestrial gravitation.[58] Further, Hooke had published his hypotheses in 1674,[47] which were evidently read by some of the leading scientists at the time, like Christian Huygens and Giovanni Domenico Cassini, who also had sent their comments to the Philosophical Transaction,[49] and he explicitly communicated his ideas directly to Newton in 1679/80. One can therefore understand Hooke's profound dismay when seven years later he first read the statement of the second law of motion as it appears in Newton's *Principia*:

"The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed···if the body moved before, is added to or subtracted from the former motion···so as to produce a new motion compounded from the determination of both···" (Ref. 59).

In similar words, these are the principles of dynamics which Hooke formulated in his May 1666 lecture to the Royal Society, and restated in his book entitled *An Attempt to Prove the Motion of the Earth* (Ref. 47) in 1674. Later on Hooke noted in his diary[60] on February 15, 1689, that

"···At Hallys [Halley] met Newton; vainly pretended claim yet acknowledged my information. Interest has no conscience: A posse ad esse non valet consequentia (It does not merit taking any steps)···"

In a letter to Newton written on January 6, 1679/80, Hooke had mentioned, without giving any supporting arguments, that

"···my supposition is that the Attraction always is in a duplicate proportion to the Distance from the center Reciprocall···it truly makes out all the Appearances of the Heaven···not that I believe there really is such an attraction to the very Center of the Earth···"

In fact he had conjectured correctly on physical grounds that

"···I rather Conceive that the more the body approaches the Center, the less it will be urged by the attraction—possible somewhat like the Gravitation on a pendulum or a body moved in a Concave Sphere (Ref. 61) where the power Continually Decreases the neerer the body inclines to a horizontal motion···"

However, in spite of Hooke's caveat, these remarks were also forgotten later by Newton,[25] who incorrectly charged

"··· that what he (Hooke) told me of the duplicate proportion was erroneous, namely that it reached down from hence to the center of the earth···"

In Newton's letters to Halley[25] in the summer of 1686, which were intended primarily to deny Hooke any credit for the ideas incorporated in the foundations of the *Principia*, Newton nevertheless admitted that Hooke's correspondence stimulated him to consider again the fundamental problems of celestial mechanics. After he had been calmed down (by a soothing letter from Halley[25]) in his bitter invective against Hooke, apparently initiated by rumors that Hooke had accused him of plagiarism, he wrote to Halley on July 14, 1686 that

"···This is true, that his Letters occasioned my finding the method of determining Figures which when I had tried in the Ellipsis, I threw the calculation by being upon other studies and so it rested for about 5 years till upon your request I sought for that paper, and not finding it did it again and reduced it into the Propositions shown you for Mr. Paget···"

and again, on July 27, 1686[25]

"···And tho his correcting my Spiral occasioned my finding the Theorem by which I afterward examined the Ellipsis; yet I am not beholden to him for any light into that business but only for the diversion he gave me from my other studies to think on these things and for his domaticalnes in writing as if he had found the motion in the Ellipsis, which inclined me to try it after I saw by what method it was to be done···"

On December 13, 1679 Newton had written to Hooke a letter that ended with

"··· Your acute Letter having put me upon considering thus far the species of this curve, I might add something about its description by points *quam proximè*. But the thing being of no great moment I rather beg your pardon for having troubled you thus far with this second scribble···"

In this letter, Newton made a drawing for an orbit of a body under the action of a constant central force, and he discussed also general properties of orbits for forces which increases toward the center. Newton's computational method "by points *quam proximè*" has been a puzzle for a long time. It has been suggested by Westfall[15] and more recently by Erlichson[62] that it was based on applying the geometrical construction in Theorem 1, Proposition 1, Book 1 of the *Principia*. However, I have given arguments[26] that this is not plausible, and that Newton's method was most likely based on his 1664–1671 work on the calculus of curvature.[63] The first time, for which there is any documentary evidence, that Newton applied to orbital motion the idea of *compounding* a tangential velocity with a radial velocity impressed by an attractive central force (which we claimed that he could not remember having heard from Hooke), was in his manuscript *De Motu Corporem Gyrum*, written four years after his correspondence with Hooke.[14] This supports the contention that Hooke contributed in a fundamental way to Newton's understanding of the dynamical principles[65] incorporated in the *Principia*. Later Newton recalled that

"In the year 1679 in answer to a letter from Dr. Hook···I found *now* that whatsoever was the law of the forces which kept the Planets in their Orbs, the areas described by a Radius drawn from then to the Sun would be proportional to the times in which they were described. And by the help of these two propositions I found that their Orbs would be such ellipses as Kepler had described···" (Ref. 11).

In 1682, Hooke elaborated this theory of universal gravitational[37] attraction in another Cutlerian lecture entitled *A Discourse of the Nature of Comets*, which was *published* only after his death.[23] His theory was that bodies emitted *periodic gravitational pulses*, in analogy with his vibrational theory of matter, sound, and light. From this *supposition*, he deduced that the intensity decreases with the inverse square of the distance from the source.

"For this power Propagated, as I shall then shew, does continually diminish according as the Orb of Propagation does continually increase, as we find the Propagations of the Media of Light and Sound also to do; as also the Propagation of Undulation upon the Superficies of Water. And from hence I conceive the Power thereof to be always reciprocal to the Area or Superficies of the Orb of Propagation, that is duplicate of the Distances (Ref. 24); as will plainly follow and appear from the consideration of the Nature thereof, and will hereafter be more plainly evinced by the Effects it Causes at such several Distances.

This propagated Pulse I take to be the Cause of the Descent of Bodies towards the Earth···Suppose for Instance there should be a 1000 of these Pulses in a Second of Time, then must the Grave body receive all those thousand impressions within the space of that Second, and a thousand more the next···" (Ref. 66).

As I have pointed out earlier, it turned out that the notion of a periodic pulsed force rather than a continuous force was essential in the mathematical formulation of Hooke's approach to orbital motion.

Then in 1684 Hooke claimed to Christopher Wren and to Edmond Halley, who were two of his friends at the Royal Society, that he could demonstrate that an inverse square law dependence of the gravitational force implied elliptic orbits for planetary motion. However, in spite of a challenge by Wren who offered a prize of a 40 shilling book for such a demonstration, Hooke apparently failed to produce a *cogent* explanation.[39] Evidently he also never published his argument. According to an account by DeMoivre in 1727, what followed is that

···in 1684 Dr. Halley made Sir Isaac a visit at Cambridge and there in a conversation the Dr. asked him what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distances from it. Sir Isaac replied immediately that it would be an *Ellipsis*. The Doctor struck with joy and amazement asked him how he knew it. Why saith he I have calculated it. Where-

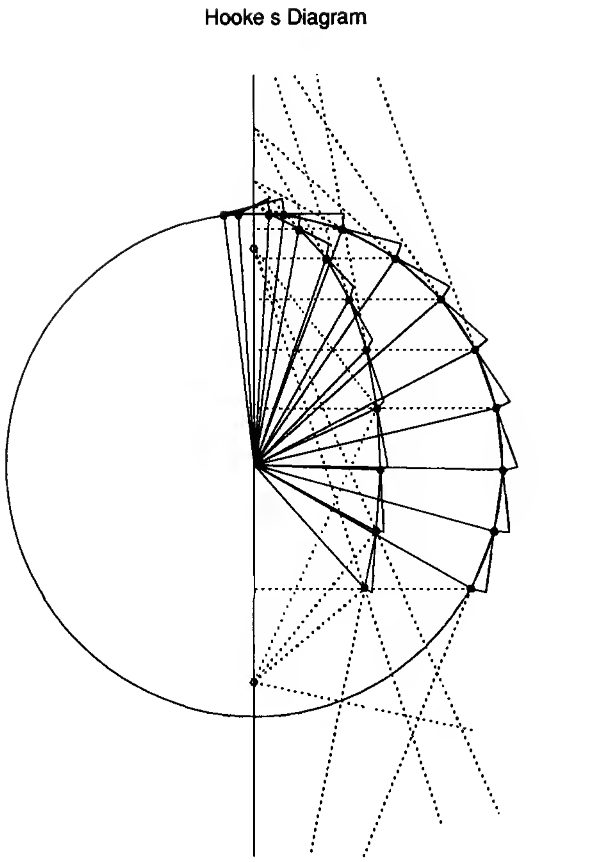

Fig. 2. Heavy lines are the result of a numerical evaluation of the orbital path following the description given in the text of Fig. 1 (see the Appendix and Sec. IV). Initial conditions where chosen to correspond approximately to those in Fig. 1. Dotted lines are additional auxiliary lines which demonstrate graphically that the orbit is an *ellipse* (see Sec. III).

> upon Dr. Halley asked him for his calculation without any further delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it, and then to send it to him..." (Ref. 25).

A couple of months later Newton sent Halley a manuscript, *De Motu*, containing his celebrated proof of the converse of the problem posed by Halley[67] that a body moving on an elliptic orbit with the center of the attractive force located at one focus implies a force with an inverse square dependence on radial distance. Newton's proof was based on a geometrical construction which embodied the dynamical principles which Hooke had directly communicated to him. Partly at Halley's urging that he publish his results, Newton then spent the next two years in further work which culminated in his monumental *Principia* which appeared in 1687.

III. HOOKE'S GRAPHICAL CONSTRUCTION

In this section, I will describe Hooke's graphical method for constructing an orbital path in a central field of force which varies linearly with the distance from the center. Following Hooke's handwritten text describing his geometrical construction, the Appendix, and transforming it into a numerical algorithm, Sec. III, Eqs. (1) and (2), I have been able to reconstruct in detail his diagram, Fig. 1, as shown in Fig. 2. In his figure Hooke draws *two* polygonal

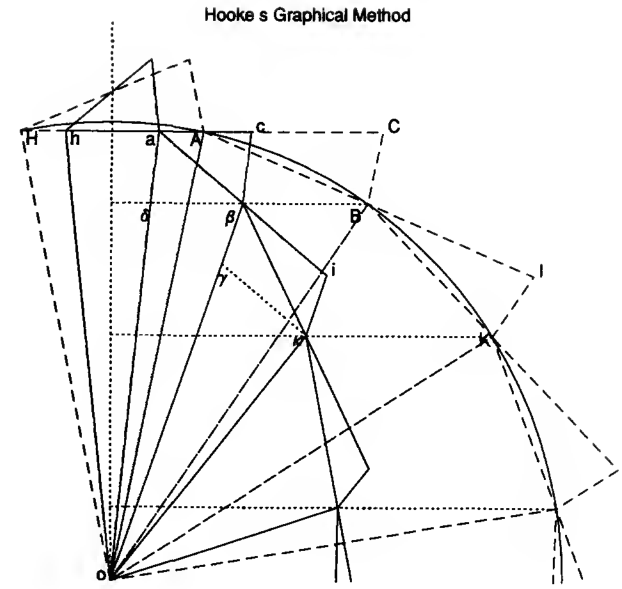

Fig. 3. Expanded version of upper right quadrant of Fig. 2 which illustrates Hooke's geometrical construction. Dashed lines correspond to a discrete circular path, and the heavy lines to a discrete elliptical path (see Sec. III).

orbits, one which he claims to be an ellipse with its vertices partly labeled by the letters h, α, β, and κ, and the other one consisting of equal chords on a circumscribed circle with unlabeled vertices.[68] In addition there are several auxiliary lines which will be discussed below. An expanded version of the first quadrant of his diagram is shown in Fig. 3, where for clarity I have also chosen a somewhat larger step size ha. The long dashed lines correspond to Hooke's graphical construction for the polygonal path on a circle with center at O, and vertices labeled by the capital letters H, A, B, K, while the full lines correspond to the construction for the polygonal path on an ellipse. As we shall see, the polygonal path on the circle serves to establish the magnitude or strength of the central attractive force pulses. For a force depending linearly on the distance, this gives a direct *graphical* way to evaluate the magnitude of the radial velocity impressed by each pulse at the end of successive equal time intervals. It should be pointed out that this is not described in Hooke's text, but can be inferred from his diagram. Further, the order in which some of the points are obtained by a purely graphical procedure is then somewhat different from that indicated in the text, which is describing a sequential order following the physical events, gravity pulses, in time.

The physical basis for Hooke's graphical construction is the replacement of a continuously varying central force by a series of instantaneous pulses varying periodically in time. Then the velocity of a particle during each short equal time interval can be obtained by adding a tangential velocity due to inertia and a radial velocity due to this impulsive central attractive force. This radial velocity is the change of velocity during the time interval and is proportional to the acceleration (see Sec. IV) due to the impressed force, although Hooke does not state explicitly that its magnitude is proportional to the *time interval* as well as to the impulse force.[69] This is not a problem if the time interval is the physical period of the impulsive force, which in Hooke's discrete dynamics is a separate parameter.

However, if the limit of increasingly smaller time intervals is examined, the dependence of the radial velocity on the period as well as the strength of the pulse must also be considered, as I will discuss later on (see Sec. IV).

To obtain the special trajectory of a polygonal orbit on a circle, Hooke considers a circle (see Fig. 3) of radius HO with center at O, and draws a horizontal chord from H intersecting the circle at A with length HA small compared to HO. This represents the (vector) distance traveled at the end of the first time interval. Actually, in the text Hooke calls such line segments velocities, which is valid provided the time interval is kept fixed. For the next time interval, he then extends this line to C with length $AC = HA$, where AC represents the distance that would be traveled during the same time interval in the absence of external forces. The effect of the pulsed central attractive force, considered to operate instantaneously at A, is obtained by drawing CB parallel to AO, intersecting the circle at B. Thus the chord AB is the resultant (vector) distance traveled during the next time interval. In this case $CB = (AB)^2/AO$. This is similar to the celebrated construction for centrifugal acceleration obtained by Newton[35,70] in 1665, and independently derived in the continuum limit by Christian Huygens[71] in 1659. Hence, iterating this construction leads to a polygonal path with vertices H,A,B,K, etc., on a circle of radius HO, corresponding to the orbit of a body under the action of a pulsed central force. This construction also fixes a parameter $\mu = CB/AO$ for the strength of the linear force (see Sec. IV).

Hooke generalizes this construction to obtain a general polygonal path in a pulsed central force field varying linearly with distance. He considers a segment ha of the chord HA, corresponding to a smaller initial tangential velocity, located symmetrically about the middle of HA, and extends the line to point c with length $ac = ha$. This point c is the position which the body would occupy during the second time interval in the absence of external forces. A line is drawn connecting point a to the origin o, and a horizontal line is drawn from point B intersecting the line ao at the point δ. This is evident from Hooke's diagram, Fig. 1, although this graphical construction is not described in the associated text, where $a\delta$ is introduced as "the velocity impress by Gravity" (see the Appendix). However, it can be readily verified that this construction gives $a\delta = \mu(ao)$, where $\mu = CB/AO$ is the force strength parameter introduced above. Indeed, for linear radial forces, the horizontal and vertical motions are completely independent of each other, and therefore the vertical motion during equal time intervals is independent of the initial *horizontal* velocity (for details see Sec. IV). The point β is determined on the line δB, by setting $\delta\beta = ac$, and the diagonal $a\beta$ gives the resulting displacement during the second time interval. This construction "make(s) ($\delta\beta$) parallell and equall to (ac)," and gives the position β at the end of the second time interval due to compounding the tangential velocity ha and the radial velocity $a\delta$. The next step is to determine the velocity $\beta\gamma$ along the radial direction βo due to the "second puls of gravity." This can be achieved graphically by taking Hooke's third step first, i.e., extending the line $a\beta$ to i, with $\beta i = a\beta$, and then drawing a horizontal line through the vertex K. A line from i parallel to βo intersects this horizontal line at κ, and a line from κ parallel to βi intersects βo at the desired point γ. Thus, the segment $\beta\kappa$ is the displacement during the third time interval. This is the graphical implementation of the requirement that the impulsive change of velocity be proportional to the distance from the center, and that these impulses occur at equal time intervals. In the text Hooke describes how to evaluate κ graphically assuming that γ has been obtained previously by the condition that "the velocity $\beta\gamma$ has the same proportion to the radius βo that $a\delta$ has to ao."

It can been verified from this construction that the triangle $a\beta o$ has the same proportion to the triangle ABO as the triangle $\beta\kappa o$ to the triangle BKO, as Hooke stated in the text, see the Appendix. Since the triangles ABO and BKO are equal, it follows that the area $a\beta o$=area $\beta\kappa o$. This is Hooke's own proof (different from Newton's) of the validity of Kepler's area law for the case of a central force varying linearly with distance, concluding that

> "···The motion of this body therefore shall be polygonall in an ellipse, and shall Describe equall areas in equall times···"

Hooke's diagram also shows at least three separate graphical tests, detailed below, which demonstrate that the vertices of this polygonal path lie on an ellipse. These tests can be deduced from auxiliary lines in the figure which are not directly related to the graphical construction described in the text. The polygonal path and its associated ellipse can also be obtained by an affine transformation of the corresponding polygonal path associated with the circumscribed circle. Since the polygonal path associated with the circle consists of equal chords of the circle, the affine transformation (see Sec. IV) gives a rigorous mathematical proof,[33] that Hooke's geometrical construction leads to a polygonal path with vertices on an ellipse.

(1) The shortest distance from the center to a segment on the polygonal path gives approximately the minor axes of the ellipse. Then the major and minor axis of the ellipse determine numerically the location of the two foci, and the sum of the distances from the foci to each vertex of the path can be evaluated. Two dashed lines from a point on the path to the foci are clearly shown in Fig. 1, which contains also two such lines from another point on the path. For the nine points which I calculated using Hooke's algorithm, I find that the sum of these distances to the foci is a constant to one part in a thousand. These errors are due to the errors in the location of the foci, which is evident in Hooke's diagram, Fig. 1.[72] The circles in Figs. 2 and 4 identify points on an ellipse with the calculated foci, in excellent agreement with the vertices of the polygonal path obtained by iterating Hooke's graphical construction.

(2) In the upper-right-hand quadrant of Fig. 1, Hooke extends the line connecting two adjacent vertices of the polygonal elliptical path and the line associated with corresponding vertices of the polygonal circular path. He finds graphically that all such pairs of lines intersect on the vertical axis through the origin. This is an obvious consequence of the affine transformation, which relates these pairs of lines.

(3) In the lower-right-hand quadrant of Fig. 1, Hooke draws an isosceles triangle. The upper vertex is located at a vertex of the polygonal path, and the lower left vertex is at the lower focus of the ellipse. The lower right vertex is obtained by extending the line from the upper focus of the ellipse to the upper vertex of the triangle by an amount equal to the distance between the upper vertex of this triangle and the lower focus of the ellipse. This is the start of a graphical method, already well known in the 17th cen-

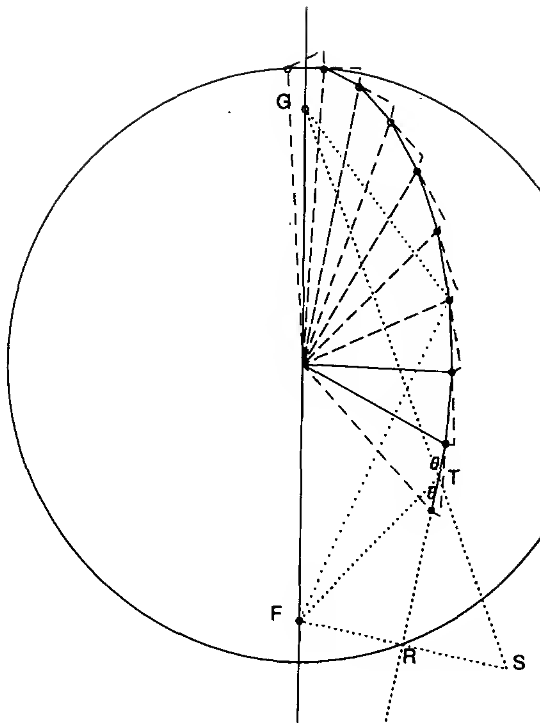

Fig. 4. Comparison of the discrete path, in heavy lines, for a linear central force, and the corresponding ellipse at discrete points located at the center of the circles. The foci of the ellipse are at F and G, and two of the lines from the foci to the path are indicated by dashed lines. For a discussion of the isosceles triangle at the bottom of the figure see Sec. III.

tury, to obtain the tangent line to an ellipse. The tangent is obtained by drawing the line passing through this upper vertex which bisects the base of the triangle (drawn poorly by Hooke), which is then extended until it intersects the vertical axis. A second line extended to this axis is shown nearly tangent to the circle at the same height. If the polygonal path obtained by Hooke's geometrical construction lies on an ellipse, the tangent line to the circle should intersect the axis at the same point as the tangent line to the ellipse. This is indeed approximately the case as shown in Fig. 1, where the small errors in Hooke's diagram can be traced to his error in the graphical location of the foci of the ellipse. A second isosceles triangle is similarly started but not completed in the bottom of Fig. 1.

A corresponding construction for the polygonal path is shown in Fig. 4, where the isoceles triangle FST is obtained by drawing a line extending from the upper focus G through the midpoint T between two vertices of the polygonal path with $TS=TF$. The extension of the line connecting these vertices bisects the base FS of the triangle at R. As can be shown analytically, R is also the intersection of the circle with FS, which is approximately the case in Hooke's diagram, Fig. 1.

It is clear that Hooke had a very sophisticated knowledge of geometrical properties of an ellipse.[73] It is possible that Hooke may have drawn the polygonal path on an ellipse in his diagram by effectively applying an affine transformation[33] to the polygonal path on the circumscribed circle, rather than by following the equivalent geometrical construction described in his text. However, neither in the text or elsewhere in the related pages of the manuscript at the Wren Library does Hooke refer explicitly to such a transformation. Regardless of whether Hooke understood fully this mathematical equivalence, his assertion that "···the motion of this body therefore shall be polygonal in an ellipse···" is strictly valid when the pulsed central force varies linearly with distance.

IV. DISCRETE EQUATIONS OF MOTION

The basic geometrical construction of Newton and Hooke can be expressed in an equivalent algebraic form by applying the analytic geometry of Descartes and Fermat, which was fundamental to the further mathematical development of mechanics. In modern vector notation, we denote the position vector by $\mathbf{r}$, and the velocity vector by $\mathbf{v}$ at a time t. Referring to Hooke's geometric construction, Fig. 3, $\mathbf{r}$ represents, e.g., the initial radius vector aO and $\mathbf{v}\delta t$ the displacement vector $a\beta$, where δt is a fixed time interval. Then

$$\mathbf{r}'=\mathbf{r}+\mathbf{v}\delta t, \tag{1}$$

where $\mathbf{r}'$ is the displacement βO, corresponding to the position vector at the end of the first time interval δt. In the absence of an external force (called power by Hooke), the velocity v is a constant, which is the mathematical expression for the principle of inertia.[43] Assuming that a central force acts by pulses applied periodically at intervals of length δt, and that its effect is to *compound* or add to $\mathbf{v}$ an *impressed* velocity vector in the direction of $\mathbf{r}'$, the $\mathbf{v}'$ after the impulse is given by

$$\mathbf{v}'=\mathbf{v}+f(r')\mathbf{r}'\delta t. \tag{2}$$

In the case considered by Hooke, corresponding to an attractive central force which varies linearly with distance, $f(r')$ is a negative constant. For example, in Fig. 3 $f(r')\mathbf{r}'\delta t^2$ represents the displacement $i\kappa$, and $\mathbf{v}'\delta t$ represent the displacement $\beta\kappa$. More generally, the function $rf(r)$ determines the dependence of the periodic impulsive force on the radial distance r, and is evaluated at the end of a time interval δt (Ref. 74) to determine the change of velocity in the subsequent time interval. This condition is important because it leads to Kepler's celebrated area law (angular momentum conservation) for central forces, applied here to the *discrete* orbital equations of motion, Eqs. (1) and (2),

$$\mathbf{r}'\times\mathbf{v}'=\mathbf{r}\times\mathbf{v}. \tag{3}$$

This property was demonstrated by Hooke for central forces which vary linearly with distance in the text associated with his diagram, Fig. 1 (see the Appendix), by a different method from Newton's proof in the *Principia*.

Given initial values of $\mathbf{r}$ and $\mathbf{v}$, *iterating* the discrete equations of motion, Eqs. (1) and (2), correspond precisely to the graphical method applied by Hooke, as described in Sec. III. However, this algebraic form has the advantage of allowing much greater numerical accuracy, and most importantly, the analytic application of the calculus, corresponding to the limit of small time steps, which leads to the well-known *differential* equations of motion.

In Eq. (2), it has been assumed that the change in velocity $\delta\mathbf{v}=\mathbf{v}'-\mathbf{v}$ is not only proportional to the force, but it is also proportional to the time interval δt, although this important point was not explicitly stated by Hooke.[69] However, it is clear from Hooke's graphical construction

that if the initial displacement to obtain a circular orbit is decreased by an arbitrary factor, the change of velocity must be decreased by this same factor. Further, the graphical construction implies that the same scaling occurs for an orbit with arbitrary initial conditions. Thus, the velocity change δv must also be proportional to δt, and in the limit of small δt the radial force is proportional to the *acceleration*. By relating this time interval to the corresponding increase in area according to Kepler's area law, Newton was able to prove theorems in the *Principia* in geometrical form. In the limit of vanishing δt, Eq. (2) becomes the differential equation

$$\mathbf{a}=f(r)\mathbf{r}, \tag{4}$$

where $\mathbf{a}=\delta\mathbf{v}/\delta t$ is the acceleration. Identifying $f(r)\mathbf{r}$ with the force per unit mass, Eq. (4) corresponds to Newton's differential equation of motion.

In the *De Motu*, and at the start of the *Principia*, Newton proceeds to apply this geometrical construction in a manner quite different from Hooke. He considers the limit when δt becomes vanishingly small. In this limit he solves the *converse* of the problem treated by Hooke: given an elliptical orbit with a force directed toward the *center* of the ellipse, he proves that the radial dependence of the force is linear with the distance from the center, Proposition X, Problem V, Book I of the *Principia*. Then in Section III, Proposition XI, Problem VI, Newton applies his method to the central problem of celestial mechanics:

> "If a body revolves in an ellipse; it is required to find the law of the centripetal force tending to the focus of the ellipse."

He proves that this force is inverse to the square of the distance to this focus.[67] Later in the *Principia*, in Proposition XLI, Problem XXVIII, he treats the general problem,

> "Supposing a central force of any kind, and granting the quadratures of curvilinear figures; it is required to find as well the curves in which bodies will move, as the times of their motions in the curves found."

Surprisingly,[26] Newton does not evaluate the integral for the main case of interest, the gravitational force, but instead he evaluates in corollary III the integral for an $1/r^3$ force, leaving the gravitational force presumably as an exercise for the reader.[75] The integral for the $1/r^2$ force was first published in 1710 by Johann Bernoulli.[76]

In closing this section, I will consider some applications of the discrete equations of motion, Eqs. (1) and (2), relevant to our discussion. The first known analytic solution was carried out by Issac Beeckman[38] for a force giving rise to constant gravitational acceleration g. In one dimension, which we take to be the x axis, we have

$$x_{n+1}=x_n+v_n\delta t \tag{5}$$

and

$$v_{n+1}=v_n+g\delta t, \tag{6}$$

where x_n is the position and v_n is the velocity after the nth iteration. Starting from rest, $v_0=0$ at $x_0=0$, one obtains from Eq. (6) that $v_n=gn\delta t$, and substituting this result in Eq. (5), one finds that

$$x_n=\frac{n(n-1)}{2}\,g(\delta t)^2. \tag{7}$$

Setting the elapsed time $t=n\delta t$ gives the velocity $v=gt$ and the position $x=(1/2)g(1-1/n)t^2$. In the limit that t is kept fixed, and the number of pulses n becomes infinitely large, one recovers Galileo's familiar result that $x=(1/2)gt^2$.

For the special case considered by Hooke, that the force varies linearly with the distance, $f(r)=-\mu$, the vector components of Eqs. (1) and (2) are not coupled. In this case these equations are invariant under the affine transformation $x'=\lambda_1 x$, $y'=\lambda_2 y$, and $z'=\lambda_3 z$, where λ_1, λ_2, and λ_3 are constants. It follows that all the bound solutions can be obtained by an affine transformation of the special polygonal path with vertices on a circle. Consequently, all bound solutions correspond to polygonal paths on an ellipse, because this figure is the affine transform of a circle. This provides rigorous justification for Hooke's assertion in the text associated with his diagram, Fig. 1, that the polygonal path lies on an ellipse, which is *uniquely* determined by the initial conditions, position, and velocity. From Fig. 1, which is taken to be in the x–y plane, one obtains $\lambda_1=1/2$ and $\lambda_2=1$.[33]

However, this proof cannot be generalized to other radial dependences of the force field. For this reason a mathematical demonstration is given here in algebraic form which is analogous to Newton's geometrical demonstration that an elliptic orbit due to a force directed toward the center of the ellipse implies that the force varies linearly with distance. This proof is problem 2 in *De Motu*, which in the *Principia* becomes Proposition X, Problem V, and it is strictly valid only in the limit of vanishingly small time steps. In Cartesian coordinates x and y, the equation for an ellipse has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{8}$$

where a and b are the major and minor axis of the ellipse, respectively. Considering two nearby points with coordinates (x,y) and (x',y'), where $\delta x=x'-x$ and $\delta y=y'-y$ can be considered as small quantities, and neglecting second-order terms in these quantities, we have

$$\frac{x\delta x}{a^2}+\frac{y\delta y}{b^2}=0. \tag{9}$$

The first equation of motion, Eq. (1), defines the velocities $v_x=\delta x/\delta t$, and $v_y=\delta y/\delta t$, and substituting these in the equation for the conservation of angular momentum, Eq. (3), we have

$$x\delta y-y\delta x=l\delta t, \tag{10}$$

where l is the magnitude of the angular momentum. Solving simultaneously Eqs. (9) and (10), we obtain

$$\delta x=v_x\delta t=-\frac{l}{b^2}\,y\delta t \tag{11}$$

and

$$\delta y=v_y\delta t=\frac{l}{a^2}\,x\delta t. \tag{12}$$

It can be readily verified that a line in the direction of $\mathbf{v}$, Eqs. (11) and (12), which is a tangent to the ellipse, crosses the x axis at a^2/x *independently* of the value of b. This is one of the properties that Hooke checked graphically, choosing the corresponding circle $b=a$ for compar-

ison with his calculated orbit, Fig. 1. The change in velocity $\mathbf{v}$ during a time interval δt can be obtained from these equations without further approximations, and we get

$$v_x'-v_x=-\frac{l}{b^2}\,\delta y=-\frac{l^2}{a^2b^2}\,x\delta t, \tag{13}$$

$$v_y'-v_y=\frac{l}{a^2}\,\delta x=-\frac{l^2}{a^2b^2}\,y\delta t. \tag{14}$$

This expression agrees with the discrete equation of motion, Eq. (2), for a force which depends linearly on distance for which $f(r)=-\mu$, where $\mu=l^2/(a^2b^2)$.

The time dependence of the position and velocity coordinates is determined by the geometrical condition, Eq. (10), that equal areas are swept by the radius vector in equal times. This condition can be applied in a straightforward fashion to calculate the period T of the orbit which is related to the area A of the ellipse by the condition

$$T=2\pi\frac{ab}{l}. \tag{15}$$

Substituting the relation $l=ab\surd(\mu)$, we obtain in this case

$$T=\frac{2\pi}{\surd(\mu)} \tag{16}$$

which shows that the period is a constant independent of the parameters of the ellipse. For completeness, one can obtain the time dependence by introducing a parametric representation of the ellipse which satisfies automatically Eq. (8)

$$x=a\cos(\theta), \tag{17}$$

$$y=b\cos(\theta), \tag{18}$$

where θ is an undetermined function of time. Expanding to first order in the change $\delta\theta$ of two nearby points on the ellipse one obtains

$$\delta x=-a\sin(\theta)\delta\theta, \tag{19}$$

$$\delta y=b\cos(\theta)\delta\theta. \tag{20}$$

Substituting Eqs. (19) and (20) in Eq. (10) for the conservation of angular momentum gives the familiar result

$$\delta\theta=\frac{l}{ab}\,\delta t=\sqrt{\mu}\delta t \tag{21}$$

corresponding to harmonic oscillations with constant frequency $\surd\mu$.

The total energy E, defined by

$$E=\frac{1}{2}\,[v_x^2+v_y^2+\mu(x^2+y^2)] \tag{22}$$

can be evaluated by substituting Eqs. (11) and (12) for the velocities in Eq. (15), and we find that

$$E=\frac{l^2}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \tag{23}$$

which proves that energy is conserved in this approximation.

In the case that the center of attraction is at one of the foci of the ellipse, which is the case of interest for celestial mechanics corresponding to Kepler's first law, this method of proof can be readily applied by taking the origin of the coordinates at this focus. We then replace Eq. (8) by

$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1, \tag{24}$$

where $c=\surd(a^2-b^2)$ is the distance of the foci to the center of the ellipse. Considering two nearby points and neglecting second-order quantities, we obtain

$$\frac{(x+c)v_x}{a^2}+\frac{yv_y}{b^2}=0. \tag{25}$$

Together with Eq. (10) for the conservation of angular momentum, Eq. (18) can be solved for the velocity components v_x and v_y

$$v_x=-\frac{a^2l}{b^2(b^2-cx)}\,y, \tag{26}$$

$$v_y=\frac{l(x+c)}{(b^2-cx)}. \tag{27}$$

From Eqs. (19) and (20) it is straightforward to evaluate the first-order changes in velocity in terms of the corresponding changes of position in the time interval δt, and one find that

$$v_x'-v_x\approx-\frac{a^4l^2}{b^2(b^2-cx)^3}\,x\delta t \tag{28}$$

and

$$v_y'-v_y\approx-\frac{a^4l^2}{b^2(b^2-cx)^3}\,y\delta t. \tag{29}$$

Recognizing that the radial distance is

$$r=(b^2-cx)/a \tag{30}$$

and substituting Eq. (23) in Eqs. (21) and (22), we obtain agreement with the discrete equation of motion, Eq. (2), with $f(r)=-\alpha/r^3$ and $\alpha=al^2/b^2$. Substituting this relation in the expression for the period T, Eq. (15), gives

$$T=2\pi\frac{a^{3/2}}{\surd\alpha} \tag{31}$$

which is the general form of Kepler's law for elliptic motion. As before, to obtain the time dependence we introduce a parametric representation for the ellipse, which takes the form

$$x=-c+a\cos(\theta), \tag{32}$$

$$y=b\sin(\theta). \tag{33}$$

In this case substituting up to first order changes in Eq. (10) implies that

$$\left(1-\frac{c}{a}\cos(\theta)\right)\delta\theta=\frac{l}{ab}\,\delta t. \tag{34}$$

This gives an implicit solution

$$\theta-e\sin(\theta)=2\pi\frac{t}{T} \tag{35}$$

which is known as Kepler's problem, where $e=c/a$ is the eccentricity of the ellipse.

The total energy E is obtained by evaluating

$$E=\frac{1}{2}(v_x^2+v_y^2)-\frac{\alpha}{r}. \tag{36}$$

Substituting Eqs. (19), (20), and (23) for v_x, v_y, and r, respectively, in Eq. (24), one obtains conservation of energy in this first order approximation,

$$E=-\frac{\alpha}{2a}. \tag{37}$$

V. APPLICATION TO THE GRAVITATIONAL FORCE

Hooke's pulsed theory of gravitation, which he discussed in detail in 1682,[37] implied an inverse square dependence on the distance for the gravitational force. Further, as was mentioned before, Halley and Wren also believed that this was the correct form of the gravitational force, and Hooke had been challenged to demonstrate that this force law implied elliptic planetary orbits.[39] Therefore it is hard to believe that Hooke would not have tried to apply his graphical method to this crucially important case. Indeed, one finds that in Hooke's 1685 manuscript[30] at the Wren Library there is a brief mention of his theory of gravitation where "···The power (force) in O are Reciprocall to the squares of the Distances···" However, I did not find a corresponding graphical construction associated with the inverse square law of force. It should be kept in mind that according to his early biographer, Richard Waller,[22] many of Hooke's manuscripts have been lost.

It is possible to construct such a diagram by Hooke's graphical method. All that needs to be changed in the geometrical construction, Figs. 1–3, is to substitute, e.g., for the condition that β and κ lie on a horizontal line with B and K, the requirement that $c\beta=a\delta$ and $i\kappa=\beta\gamma$ are proportional to the inverse square of the lengths ao and βo, respectively. The corresponding algebraic construction is obtained by iterating Eqs. (1) and (2) with $f(r)=-\alpha/r^3$, where α is a constant of proportionality (see Sec. IV). I assume initial conditions and a step size comparable to those in Hooke's diagram, Fig. 1. The result of this calculation for eight iterations is shown in Fig. 5. Until the sixth iteration the agreement with an elliptic path, which is shown at discrete points by the small open circles at the angular coordinates corresponding to the discrete path, is very good. However a noticeable deviation occurs after the seventh iteration, and the discrete trajectory begins to diverge strongly from an elliptic path near the center of force as shown by the result obtained at the eight iteration. If Hooke carried out such a calculation graphically, as I strongly suspect he would have done, this result must have presented him with a very serious dilemma. He would have realized that his *graphical* method worked very well and gave an elliptic orbit provided the path did not approach too close to the force center where the inverse square law force becomes divergent. This is not a problem for planetary orbits which have a very small eccentricity, like the planets around the sun. However, it is not clear how he could have overcome this difficulty which is present for very eccentric orbits like those of comets. Hooke was aware that a continuous path was obtained by taking smaller and smaller time steps.[30] For example, if he had taken half the initial step size he would have obtained the improved result shown in Fig. 6, provided he scaled the radial displacement appropriately by a factor 1/4. However by then he would have reached the limit of accuracy which seems possible by a graphical construction. At this point it is not clear whether he would have realized that the only way that he could have carried the iterations further with smaller time steps would be by transforming the graphical construction into a numerical algorithm as indicated in Eqs. (1) and (2). This step was open to him by Descartes' and Fermat's introduction of analytic or alge-

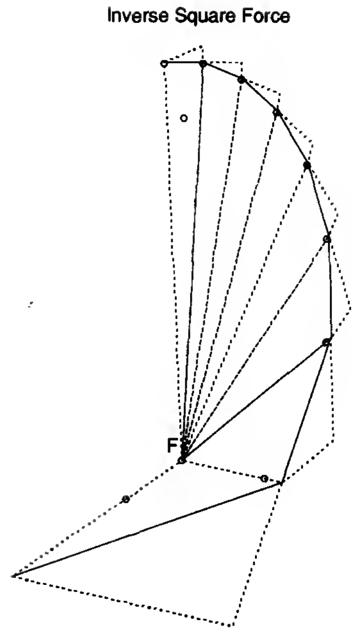

Fig. 5. Discrete orbit obtained from the geometrical construction of Hooke and Newton (see Sec. V) for the case that the central force varies inversely with the square of the distance from the center labeled F. The initial conditions and step size are similar to those in Hooke's diagram, Fig. 1. The center of the circles locate the corresponding elliptical orbit.

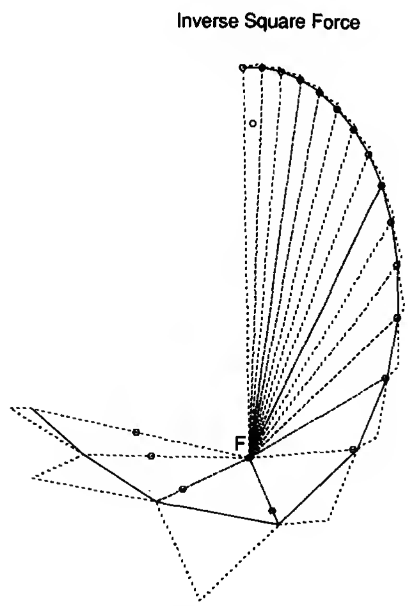

Fig. 6. The same calculation as in Fig. 5 with half the time step.

braic geometry, but there is no evidence that he availed himself of this important mathematical development.

VI. CONCLUSIONS

The analysis discussed here of the 1685 diagram of Robert Hooke shown in Fig. 1, describing a geometrical construction and graphical evaluation of the path of a body in a radial field of force, gives new evidence that Hooke had come much closer to a *mathematical* formulation of his principles of dynamics than has been previously thought. He had developed these principles correctly in the middle 1660s, demonstrating them with *mechanical analogs* like the conical pendulum, and balls rolling on various surfaces of revolution. Hooke was not simply *guessing* these principles, as Newton and many subsequent historians of sciences have assumed, but he applied precisely the same *rules of reasoning* which Newton later included at the start of Book III of the *Principia*. However, until the recent publication of Hooke's diagram,[29] Fig. 1, there had not been any concrete evidence concerning the extent to which Hooke was able to formulate his dynamical principles in mathematical form, and to apply them to the evaluation of orbital motion in a central field of force.

The key idea for a mathematical formulation, which had apparently eluded Hooke for such a long time, was based on the approximation that the forces are applied during instantaneous impulses which occur periodically in time rather than acting continuously in time. Hooke had conjectured that the gravitational attraction is a periodic pulse, in analogy with the emission of light and sound,[23] and applied this idea to deduce that the strength of gravity varied with distance inversely as the square of the distance. Newton's letter of December 13, 1679 indicated to Hooke that Newton was capable of evaluating, at least approximately, orbital motion in a central field of force. For a constant force Newton's diagram was in agreement with his previous observations of a mechanical analog, a ball rolling in an inverted cone. However, apart from the cryptic remark that "⋯(for I here consider motion according to the method of indivisibles)⋯", (Refs. 8, 77) it must have become clear to Hooke that he would not get any mathematical help from Newton. Knowing that a solution exists, at least for the related problem of constant radial gravity, would have given further stimulus to Hooke to find a mathematical formulation of his dynamical principles. In a letter dated June 29, 1686 to Newton,[25] Halley, recounting their meeting in August 1684, says that

> "⋯I then learnt the good news that you had brought this demonstration to perfection and you were pleased, to promise me a copy thereof, which the November following I received with a great deal of satisfaction from Mr. Paget⋯since which time it has been entered upon the Register books of the Society as all this past Mr. Hook was acquainted with it; and according to the philosophically ambitious temper he is of, he would, had he been master of a like demonstration, no longer have conceald it, the reason he told Sr Christopher and I now ceasing. But now he sais that this is but one small part of an excellent System of Nature, which he has conceived, but has not yet compleatly made out, so that he thinks not fit to publish one part without the other. But I have plainly told him, that unless he produce another differing demonstration, and let the world judge of it, neither I nor any one else can believe it⋯"

Apparently Hooke never showed Halley his quite "differing demonstration," Fig. 1, corresponding to a graphical solution of the *indirect problem*, at least for the linear force, and evidently he did not take his advice to publish it.

Whether or not Hooke had seen Newton's manuscript *De Motu*[34]—I could not find any reference to *De Motu* among Hooke's unpublished manuscripts in the Trinity library—he had indeed found a different demonstration. In *De Motu* Newton had considered what in the 18th century became known as the *direct problem* of mechanics, that is, from a given orbital path, e.g., an ellipse, to deduce the radial dependence of the force directed to either the center or to one of the foci of the ellipse. On the other hand, Hooke applied this geometrical construction for an iterative graphical evaluation of the *indirect problem* corresponding to the first of these two cases: calculating the orbital path for a given force varying linearly with distance. He found that the orbit is a polygonal path with vertices on an ellipse, which can be demonstrated to be rigorously valid (Sec. IV) by an affine transformation,[33] for finite time steps. However, to analyze the orbit for other forces it is necessary to take the limit of vanishingly small time steps, as was done by Newton. Only later in the *Principia*, did Newton discuss the indirect problem to calculate an orbit in an arbitrary central field of force, giving the angular position and time of the motion in terms of the *quadratures* of *curvilinear figures*, Proposition XLI, Problem XXVIII. However, Newton does not give the resultant integral for the primary case of physical interest, the $1/r^2$ force, which was first published by Johann Bernoulli[76] in 1710.

While a great deal of heat was generated on the question of priority concerning the discovery of the inverse square law of gravitational force, it is interesting to note that Hooke as well as Newton had also paid attention to the problem of finding the radial dependence of a central force which leads to simple closed orbits *for all initial conditions.* For example, from Hooke's experiments of a "⋯ball Rouling in an inverted cone⋯", corresponding approximately to a constant central gravitational force, and from Newton's calculations they both realized that the orbits did not necessarily close. Historians of science have often reproduced Newton's drawing erroneously as a *closed* orbit even though Newton had carefully avoided to commit himself in his figure.[26] Newton realized that if the force increased with the distance to the center, the point of closest approach O "⋯may fall in the line CD⋯",[8] which by symmetry, evidently assumed by Newton to be valid,[26] would imply a simple closed orbit. Likewise, from his diary[78] it appears that Hooke was trying various surfaces of revolution, to see if he could find closed orbits which would also be ellipses. On September 7, 1677 one finds the entry that

> "⋯The planetary attraction is expressd by a bullet Descending on the back of a cubical Paraboloeid complement⋯"

The appropriate analog surface for the gravitational force is a hyperbolic surface of revolution, but there is no direct evidence that Hooke discovered this important property, except for his remarkable sketch[9] on the resulting orbit. He left only the tantalizing entry in his diary on August 22,

1677 that he "Invented planetary Line on hyperbolicall consect the velocity about one asymptote and the planet on the other···" (Ref. 11).

It is clear that without a systematic procedure to approach the limit of indefinitely smaller time steps, i.e., the concept of the infinitesimal calculus which Newton and Leibniz helped develop, Hooke could have solved the problems of mechanics by his discrete graphical method only in an approximate manner. More importantly, he could not apply the powerful analytic tools of the differential calculus. It is interesting that among Hooke's unpublished manuscripts at the library of the Royal Society in London, one finds a portion of a handwritten translation[79] into English of the Marquis de l'Hospital's 1696 treatise entitled *Analysis of infinitesimal small quantities to describe curved lines.*[80] This is the first published textbook on *differential* calculus, based on lectures given by Bernoulli who had been hired by l'Hospital as his tutor.[81] Assuming he received a copy of this text, Hooke must have understood the importance of this mathematical development which is not discussed in much detail in Newton's *Principia*. Hooke may also have become familiar with Leibniz's original work[82] published first in 1684. Unfortunately the diaries of Hooke during the crucial period 1682–1687 have also been lost. There is also no evidence of any investigation on orbital motion from the transcripts of the Royal Society for this period.[83] Calculus was, of course, precisely the mathematical tool which Hooke had been missing, but unfortunately for him he had gone for mathematical help to Newton. It is interesting to speculate on the alternative development of orbital dynamics if in 1679 Hooke had corresponded with Leibniz or Huygens rather than with Newton.

Among Hooke's unpublished manuscripts in the Wren Library at Trinity College there is a memorandum entitled *A True state of the case and controversy between Sir Isaac Newton and Dr. Robert Hooke as to the Priority of that Noble Hypothesis of Motion of ye Planets about ye Sun as their center.*[57] In this document, Hooke recounts his main published contributions on the theory of mechanics and gravitation, and his correspondence with Newton in 1679/80, but he does not mention his unpublished 1685 graphical computation and related work. This memorandum makes it clear that Hooke understood quite well the extent and the relative importance of his contributions to celestial mechanics and to Newton's work. Contrary to the opinion of historians of science who claim that Hooke failed to get acknowledgments for his contributions because he had exclusively claimed priority for the discovery of the inverse square law of the gravitational force, Hooke's memorandum indicates that it was Newton who managed to focus the controversy on this single issue, presumably to be able to justify denying any credit to Hooke. After Hooke's death, Richard Waller commented[22] that

> "···Dr. Hook who was as I could prove were it a proper time the first Inventor or if you please first Hinter of those things about which Magni Nominis Heroes have contested for priority···" (Ref. 84.)

This enigmatic comment suggests that there may have been other relevant documents to Hooke's contributions to dynamics that are now lost.

ACKNOWLEDGMENTS

I would like to thank in particular Professor D. T. Whiteside, and I. B. Cohen for valuable comments and criticism. I would also like to acknowledge conversations concerning Newton and Hooke with my astrophysics colleague J. Faulkner, and valuable comments on this work by him and by my colleagues J. Deutsch, S. Flatte, and G. Gaspari. Finally I thank the staff of the library at Trinity College, Cambridge, and at the Royal Society, London, for helpful access to the unpublished manuscripts of Robert Hooke. The warm hospitality of Tali and Doug Lin made particularly pleasant my visit to Cambridge, where part of this work was done.

APPENDIX: HOOKE'S HANDWRITTEN TEXT

A transcription of Hooke's handwritten text in his diagram, Fig. 1, is printed below.[85] For clarity I have taken the liberty to include punctuation and capital letters whenever I consider this helpful. Words shown in brackets[···] have been crossed out in the text.

> "Let *ha* represent the impress velocity [as before] in the tangent [subtense] of an ellipse [circle] and $a\delta$ the velocity impress by Gravity. Make ($\delta\beta$) parallel and equal to (ac), then draw the diagonall ($a\beta$). The second puls of gravity shall meet the body at β where the puls againe meets it, driving it toward the center o with the velocity $\beta\gamma$ which has the same proportion to the radius βo that $a\delta$ has to ao. Make $\beta\, i = a\beta$ and make $\gamma\kappa$ equall and parallel to it, then draw $\beta\kappa$. Now if the velocity to gravity had been as *ha* to δa, then the body had moved in a circle, but because the velocity *ha* is less in proportion to $a\delta$ than it ought to make it move in a circle, therefore its motion shall be in an ellipse. For as ao is to $a\delta$, *soe* βo to $\beta\gamma$, *soe* io to $i\kappa$, etc., and the same proportion that $a\beta o$ has to abo the same shall $\beta\kappa o$ have to bko. The motion of this body therefore shall be polygonall in an ellipse, and shall Describe *equall areas in equall times.*
>
> When the velocity and direction of the motion of Lation [translation] doth by its Receding from the center ballance the accesse by the Ray of Gravity then doth the body move in a circle if the gravity be to the center of it. But if the Recesse overballance the accesse it goeth further off: and the contrary if contrary. And the polygone become various according to the differing degrees of Gravity at Differing distances from the center."

The substitution of the word *tangent* for the correct word *subtense* (chord) which Hooke crossed out, indicates that he regarded the chord as a good approximation to the tangent. In the diagram, Fig. 1, Hooke did not label the vertices associated with the polygon on the circular path. However, his reference to triangles *abo* and *bko* in this text correspond to two triangles associated with this circular path, which are the triangles labeled ABO and BKO in Fig. 3. Hooke's statement that "···as ao is to $a\delta$, *soe* βo to $\beta\gamma$···" follows from his graphical construction corresponding to a linear force (see Sec. III). However, he continues with "*soe* io to $i\kappa$" which is incorrect, because the point i is an auxiliary point obtained by extending $a\beta$ setting $\beta i = a\beta$, and therefore it does not lie on the polygonal path.

[1]I. Newton, *Philosophiae Naturalis Principia Mathematica*, Mathematical Principles of Natural Philosophy (first published in Latin in 1687), the third edition was translated into English by A. Motte in 1729, and revised by F. Cajori (University of California Press, Berkeley, 1934).

[2]E. N. da C. Andrade, *Robert Hooke*, Wilkins lecture, Proc. R. Soc. Ser. A **201**, 439–473 (1950); Sci. Am. **191**, 94–98 (1954).

[3]R. T. Gunther, *Early Science in Oxford* (Oxford University, Oxford, 1930) Vols. I–XV.

[4]M. 'Espinasse, *Robert Hooke* (William Heinemann, London, 1956).

[5]M. Hunter and S. Schaffer, *Robert Hooke, New Studies* (Boydell, Woodbridge, 1989).

[6]E. Mach, *The Science of Mechanics*, 6th ed. (Open Court, La Salle, IL, 1989), p. 230.

[7]W. W. Rouse Ball, *An Essay on Newton's Principia* (MacMillan, New York, 1893).

[8]J. Pelseneer, "Une lettre inédite de Newton," Isis **12**, 237–254 (1929).

[9]A. Koyré, "An unpublished letter of Robert Hooke to Isaac Newton," Isis **43**, 312–337 (1952); *Newtonian Studies* (Harvard University, Harvard, 1965).

[10]L. D. Patterson "Hooke's Gravitation Theory and its influence on Newton" (part one), Isis **40**, 327–341 (1949); "The insufficiency of the traditional estimate," Isis **41**, 32–45 (1950).

[11]J. Lohne, "Hooke versus Newton," Centaurus **7**, 6–52 (1960).

[12]D. T. Whiteside, "Newton's early thoughts on planetary motion: A fresh look," Br. J. Hist. Sci. **2**, 118 (1964); "Before the Principia: The maturing of Newton's thoughts on dynamical astronomy, 1664–1684," J. Hist. Astron. **1**, 5–19 (1970); "The mathematical principles underlying Newton's Principia Mathematica" *ibid.* **1**, 116–138 (1970).

[13]F. F. Centore, *Robert Hooke's Contribution to Mechanics, A Study in Seventeenth Century Natural Philosophy* (Martinus Nijhoff, The Hague, 1970).

[14]I. B. Cohen, *The Newtonian Revolution* (Cambridge University, Cambridge, 1980), "Newton's discovery of gravity," Sci. Am. **244**, 167–179 (1981); *The Birth of a New Physics* (Pelican, New York, 1985); *Introduction to Newton's Principia* (Cambridge University, Cambridge, 1971).

[15]R. S. Westfall, "Hooke and the law of universal gravitation," Br. J. Hist. Sci. **3**, 245–261 (1967); *Force in Newton's Physics* (Macdonald, London, 1971); *Never at Rest, A Biography of Isaac Newton* (Cambridge University, Cambridge, 1980).

[16]A. Rupert Hall, *The Revolution in Science 1500–1750* (Longman, London, 1983).

[17]R. Dugas, *Mechanics in the Seventeenth Century* (Èditions du Griffon, Neuchatel-Switzerland, 1958); *A History of Mechanics* (Neuchatel, Switzerland, 1955).

[18]E. J. Dijksterhuis, *The Mechanization of the World Picture* (Oxford University, Oxford, 1961).

[19]J. L. Heilbron, *Elements of Modern Physics* (University of California Press, Berkeley, 1982).

[20]R. Hooke, *The Sixth Cutlerian Lecture de Potentia Restitutiva* (published in 1678, Ref. 3, Vol. VIII, p. 331). In this work Hooke deciphered his anagram presented two years earlier, ceiiinosssttuv, as *ut tensio sic vis* (as the extension so the force). It is not generally recognized that Hooke was able to obtain the correct phase space relation between velocity and position for the harmonic oscillator, shown in a graph in a beautiful frontispiece of his paper. He also showed that the period of the spring was independent of amplitude, but he did not obtain the correct dependence of amplitude and velocity on time.

[21]V. I. Arnol'd, *Huygens and Barrow, Newton and Hooke* (Birkhauser, Basel, Boston, Berlin, 1990). Arnol'd claims that Hooke obtained his results on orbital motion in correspondence with Newton in 1679/80 by "···integrating equations of motion···." However, no documents have been found dated earlier than the 1685 manuscript described in this paper, which shows that Hooke had formulated his physical ideas on orbital motion in mathematical form. Arnold's suggestion that Hooke used the law of conservation of energy is insufficient to determine orbital motion in central forces, unless Hooke had applied also the law of conservation of angular momentum (Kepler's area law). However, the correspondence with Newton in 1679 shows that at that time Hooke did not yet understand Kepler's area law.

[22]Richard Waller, *The Life of Robert Hooke, The Posthumous Works of Robert Hooke*, 2nd ed. (Cass, London, 1971) pp. i–xxviii.

[23]R. Hooke, Ref. 22, pp. 167 and 179.

[24]The analogy of gravity to the emission of light from a point source, which leads to the inverse square dependence on distance, had been proposed earlier by Ismael Boulliau in 1645 (Ref. 12). Newton told Halley that Hooke had plagiarized this idea from Boulliau, and should therefore not have claimed credit for it.

[25]H. W. Turnbull, *The Correspondence of Isaac Newton* (Cambridge University, Cambridge, 1960), Vol. II.

[26]M. Nauenberg, *"Newton's Early Computational Method for Dynamics,"* Archive for History of Exact Sciences **46**, 221–251 (1994).

[27]Reference 3, Vol. VI, p. 265; T. Birch *The History of the Royal Society of London* (Royal Society, London, 1756–57), pp. 91–92.

[28]A similar dictum to Hooke's, that

"··· to the same natural effects we must, as far as possible, assign the same causes."

was enshrined four years later by Newton in the *Principia as Rule II of Rules of Reasoning* (Ref. 1); A. Koyré (Ref. 9) has pointed out that in the third edition of the *Principia* Newton added the comment "···*quatenus fieri potest* (as far as they can be)···" While Koyré did not give any reason for these changes which Newton made in consecutive editions of the *Principia* (see also footnote 1), it seems to me that there is a possibility that meanwhile Newton had read Hooke's 1682 remarkable paper on *A Discourse on the Nature of Comets* (Ref. 23), where Hooke's philosophical statements can be found. This paper was among the posthumous works of Hooke which Richard Waller dedicated to Newton in 1705. Given Newton's great interest in comets, and Hooke's excellent article on this subject, it is likely that Newton studied it carefully.

[29]P. J. Pngliese, *"Robert Hooke and the Dynamics of Motion in a Curved Path,"* in Ref. 5, pp. 181–205.

[30]I would like to thank the Master and Fellows of Trinity College, Cambridge, for permission to reprint this diagram and text from the manuscripts of Robert Hooke, MS. O.11a.1/16.

[31]Among the pages of Hooke's 1685 manuscript in the Wren Library, one finds a formulation of the *Laws of Circular Motion* which begins with the following statement:

"The Respective velocitys of Bodys moved by any powers [forces] are in subduplicate proportion of the said powers, that is as the square root of the aggregate of all the powers contributing to that motion. This is evident in all mechanical tryalls of motion as of Falling bodys, pendulums, the Running of water or any other liquors···"

In his application of this law to a constant "power" (force) and to a power varying linearly with distance, it becomes apparent that Hooke had in mind by an "aggregate of all the powers" over distance the modern concept of work or potential, and that he had formulated the law of *conservation of total energy*. Indeed he gets the correct results that

"···if the powers be equall the whole length then the velocitys shall be the ordinates of a Parabola (the modern x and y axis for a parabola were interchanged in the 17th century)···If the powers be in the same proportion with the Distances the velocities shall be as the ordinate of a quadrant of a circle."

However, it is clear that he does not know how to evaluate the aggregate (integral) for other force laws which he considers, and in particular for the gravitational case where the force varies as $1/r^2$. He starts with the supposition that

"If the powers shall be in Reciprocall proportion to the squares of the Distances the velocities shall be as"

and follows this by an empty space. Elsewhere in the manuscript he begins with the statement that

"Prove that in the access of a body to a gravitating center where the powers increase in Reciprocall proportion to the square of the Distance, the velocitys increase in reciprocall proportion to the Distance."

again followed by another empty space. It is clear that as late as 1685 he did not know how to obtain the potential for a force that varies as $1/r^2$, and therefore he could not in this case determine the dependence of velocity on distance from his law of energy conservation.

[32]D. T. Whiteside, *The Preliminary Manuscripts for Isaac Newton's 1687 Principia, 1684–1685* (Cambridge University, Cambridge, 1989).

[33]The affine transformation is $x' = \lambda x$, $y' = y$, where λ is a constant. From Hooke's diagram, Fig. 1, one finds that $\lambda = 1/2$. I am indebted to D. T. Whiteside for calling to my attention that Hooke may have applied this transformation in his diagram.

[34]Pugliese states that "···The principal argument against the precise contents of Newton's track *De Motu* (1684) being generally known to the members of the Royal Society prior to the appearance of the *Principia* is the total lack of discussion of these contents", Ref. 29. Lohne states that "···it seems that he (Hooke) only gradually became aware of the con-

tents of the work Newton was composing. But from the fall of the year 1686 we can see him feverishly active to assert his claims of priority···," Ref. 11, p. 36.

[35]J. Herivel, *The Background to Newton's Principia A Study of Newton's Dynamical Researchers in the Years 1664–84* (Oxford, University, Oxford, 1965).

[36]Galileo Galilei, *Discourse and Mathematical Demonstrations Concerning Two New Sciences Regarding Mechanics and Local Motion*, Originally published in Leiden in 1638, translated by H. Crew and A. di Salvio in 1914 (Dover, New York, 1954). For the special case of a central force that varies linearly with distance, treated by Hooke, the motion is also separable into perpendicular directions (see Sec. IV).

[37]Reference 23, p. 149.

[38]I. Beeckman, *Journal tenu par Isaac Beeckamn de 1604–1634* (La Haye, 1939), Vol. 1, p. 262; in 1618 Beeckman, the headmaster of the Latin school at Dordrecht, Holland in collaboration with Descartes, succeeded in evaluating correctly the motion of a body in a constant gravitational field, by assuming that gravitation acted in periodic pulses, Ref. 18, p. 330. It is perhaps surprising that this very fruitful idea for a mathematical treatment of orbital motion was apparently not generalized before Hooke and Newton took it up again.

[39]H. W. Turnbull, *The Correspondence of Isaac Newton* (Cambridge University, Cambridge, 1960), Vol. II, p. 442.

[40]The concept that celestial orbits could be *calculated* from physical principles was new at this time. Although Galileo had declared that "Il libro della natura é scritto in lingua matematica" (The book of nature is written in mathematical language), he did not consider celestial mechanics and the mathematical problems associated with central forces. An alternate view during that period was expressed in the remark that "The Rules of Mathematicks, or Learning by Demonstration, do ill square to Nature. For man doth not measure Nature: but she him," Ref. 19, p. 22, footnote 49.

[41]A. Armitage, "Borell's hypothesis and the rise of celestial mechanics," Ann. Sci. **5**, 342–351 (1950).

[42]The idea of the circular pendulum as a model of planetary motion seems to have originated with the astronomer J. Horrock, and it was discussed also by C. Wren, who was a close associate of Hooke. See, for example, J. A. Bennet, *The Mathematical Science of Christopher Wren* (Cambridge University, Cambridge, 1982).

[43]The first correct formulation of the principle of inertia was given by Descartes. It appeared in his book *Principles of Philosophy* first published in 1644, in which he states that "···all movement is, of itself, along straight lines; and consequently, bodies which are moving in a circle always tend to move away from the center of the circle which they are describing···." Galileo applied the law of inertia only to local motion on the surface of the earth. Like the ancients, he believed that planetary orbits were *natural*, meaning circular, and therefore one did not need a dynamical explanation based on attractive gravitational forces. Descartes' work was very influential in the seventeenth century, particularly on Newton in regard to algebraic geometry, optics, and dynamics.

[44]Hooke could have considered also the similar motion of a body at the end of an elastic string or spring, as he had discovered the law of elasticity (Ref. 20). However, no evidence has been found that he experimented with this system.

[45]For the circular pendulum, the linear dependence of the force on the distance of the suspended weight from the axis is only an approximation. This accounts for the precession of the axis of the projected ellipse which was observed by Hooke. Hooke realized that an important inadequacy of this model for planetary motion was that the center of force is at the center of the ellipse, and not at one of the foci, and experimented with the motion of balls on various surfaces of revolution. Thus, when he received Newton's letter with a sketch of the orbit of a body in a constant central field he immediately responded that "···your calculation of the curve by a body attracted by an equall power at all distances from the center Such as that of a ball Rouling in an inverted concave cone is right and the two auges [apsides] will not unite by about a third of a revolution···." Ironically, while Newton's drawing of an orbit in a constant field is strictly incorrect (see Ref. 6), as Pelseneer already pointed out in Ref. 8, it is indeed a possible orbit in an inverted cone.

[46]Reference 3, Vol. VI, p. 256.

[47]R. Hooke, *An Attempt to Prove the Motion of the Earth from Observations* (London, Printed by T. R. for John Martyn Printer to the Royal Society at the Bell in St. Pauls Church-yard, 1674). This is reprinted in Ref. 3, Vol. VII, pp. 1–28. The idea to demonstrate the motion of the earth by observing the parallax of stars can be traced back to the ancient Greeks.

[48]Reference 3, Vol. VI, p. 343.

[49]An account of Hooke's book appeared in *The Philosophical Transactions*, IX, **101**, 12, (1674) (published by the then foreign Secretary of the Royal Society, H. Oldenburg). Four issues later there appeared extracts of letters of comment by Christian Huygens, the most eminent 17th century scientist on the continent, and by the Director of the Paris observatory, Giovanni Domenico Cassini. It is evident from Newton's correspondence with Hooke and Halley that he had also become familiar with this book. This book has been reprinted by Gunther, Ref. 3, Vol. VIII, pp. 1–28.

[50]Hooke started his research work as an assistant to Boyle. It was Hooke and not Boyle who first discovered or confirmed the relation between the pressure and the volume of a gas which has become known as Boyle's law, a fact which even Newton acknowledged; see I. B. Cohen, "Newton, Hooke and Boyle's Law," Nature **204**, 618–621, (1964).

[51]Reference 3, Vol. VIII, pp. 217–271.

[52]E. Halley, *Astronomical Tables* (London, MDCCLII). In a Synopsis of the astronomy of comets, Halley credits the philosopher Seneca with placing comets "···amongst the celestial bodies···," predicting that "···there should be ages sometimes hereafter, to whom time and diligence should unfold all these mysteries···" The prevalent opinion, until Kepler's observations of the comets parallax, was that these objects were "···below the moon···." Although Hooke had lectured on comets several times at meetings of the Royal Society, demonstrating great insight into the nature of comets, he is not mentioned in this introduction.

[53]S. Pepys, *Diary* (Wheatley–Bell, London, 1904), Vol. 4, p. 341.

[54]It turns out that Hooke conjectured correctly that the light from the sun is responsible, through radiation pressure, for the formation of a component of the tail of the comet which is made up of dust.

[55]In Hooke's diary on September 20, 1679, "···Discoursed with him [Wren] about *o* [solar] Theory. He affirmed that if the motion were reciprocall to the distance the Degree of velocity should always be as the *areas*, the curve whatever it will···" This remark shows that Hooke became aware of Kepler's area law in his conversations with Wren. However, it indicates also confusion with a dependence of the velocity with the inverse of the radial distance, which is only valid at the apsides.

[56]At this time, Hooke's understanding of dynamics was based on his mechanical analogs rather than on mathematical reasoning. This important point has been missed by historians of science as is illustrated, for example, by Whiteside's comment that Hooke "··· could only compliantly answer···" Newton's letter, *The Mathematical Papers of Isaac Newton 1684–1691*, edited by D. T. Whiteside (Cambridge University, Cambridge, 1974), Vol. VI, p. 12. Although Newton's figure for a body moving in a central constant force has a substantial error in the angle between apsides, it turns out that it does correspond to a possible orbit for a ball rolling inside an inverted cone, when projected on a plane normal to the cone's axis.

[57]R. Hooke, *A True State of the Case and Controversy Between Sir Isaac Newton and Dr. Robert Hooke as the Priority of that Noble Hypothesis of Motion of the Planets About the Sun as Their Centre.* This is an undated manuscript in the Trinity College library in Cambridge, which has been reproduced by Gunther, Ref. 3, Vol. X, pp. 57–60.

[58]Concerning Newton's early ideas about the *cause* of gravity, an example is recorded in T. Birch, Hist. R. Soc. London, **3**, 248–260 (1756–57). In 1675 Newton speculated, "···after experimenting with bits of paper which are attracted to a piece of glass after rubbing it··· with some rough and raking stuff···," that

> "···the gravitating attraction of the earth be caused by the continual condensation of some other such like aethereal spirit, not of the main body of phlegmatic aether, but to something very thinly and subtilly diffused through it, perhaps of an unctuous or gummy, tenacious and springy nature, and bearing much the same relation to aether, which the vital aereal spirit, requisite for the conservation of flame and vital motions does to air···"

On February 28, 1678/9, Newton wrote a lengthy letter to R. Boyle (Ref. 25), in which he mentions near the end his latest ideas about the origins of gravity:

> "I shall set down one conjecture more which came into my mind now as I was writing this letter. It is about the cause of gravity. For this end I will suppose ether to consist of parts differing from one another in subtilty by indefinite degrees···"

It is worthwhile to note that this idea, attributed to Borell, had been

discussed and dismissed by Hooke more than a decade earlier, in his May 1666 lecture to the Royal Society, Ref. 27.

[59]It should be pointed out that in the *Principia*, Newton carefully defined quantity of motion as "···velocity and quantity of matter conjointly," which is the *momentum*. This important concept which introduced the role of *mass* was not made by Hooke. In the *Principia*, Newton does not appear to claim credit for any of the three *Laws of Motion*, stating in a subsequent Scholium that

"Hitherto I have laid down such principles as *have been received* by mathematicians, and are confirmed by abundance of experiments. By the first two Laws and the first two Corollaries, *Galileo* discovered that ··· However, Newton was the first who understood that the three laws of motion constituted the foundations of classical dynamics on which the *Principia* was erected.

[60]Reference 3, Vol. X, p. 98

[61]Here Hooke demonstrates again his considerable understanding of the concept of an *impressed* force. He points out the equivalence (in the sense of giving the same dynamical effects) of the force due to the tension of the string suspending the weight of a circular pendulum, and the force (always normal to the surface if one neglects friction) on a ball rolling inside a hollow sphere.

[62]H. Erlichson, "Newton's 1679/80 solution of the constant gravity problem," Am. J. Phys. **59**, 728–733 (1991).

[63]*The Mathematical Papers of Isaac Newton 1664-1666*, edited by D. T. Whiteside (Cambridge University, Cambridge, 1967), Vol. I, pp. 252–255 and 1670–1673 (Cambridge University, Cambridge, 1969), Vol. III, pp. 151–159.

[64]There exists an undated manuscript, MS. VIII [see J. Herivel, *The Background to Newton's Principia* (Oxford University, Oxford, 1965), pp. 246–256], where Newton gave a proof based on curvature ideas that elliptic orbits with the force acting towards a focus imply the inverse square force law. In 1690, John Locke received an altered copy of this manuscript after he had asked Newton for help in understanding the *Principia*. (It is said that eventually Locke gave up, and was content to accept Huygen's word that the mathematics of Newton was correct. This was probably the earliest example of the split which has developed between philosophy and mathematical science.) Herivel proposed that MS. VII was the "original" 1679 proof which Newton attained immediately after breaking off his correspondence with Hooke, and assertion later taken up by Westfall, while Whiteside suggested that MS VIII is an offspring of a variant proof of the 1684 manuscript *De Motu*. Recently, J. B. Brackenridge has argued in an article entitled *The Locke/Newton Manuscripts Revisited: Conjugates, Curvatures and Conjectures*, Archives Internationales D'Histoire des Sciences (to be published), that MS VIII cannot be related to Newton's 1679 solution. See also Ref. 26.

[65]Even as late as 1684, when he composed his tract *De Motu*, Newton calls inertia a *force, vim corporis* (which was the prevalent view in the middle Ages before Galileo), in the same vein that he defines an impressed force as *vim centripetam* (centripetal force), in Definition 1, Ref. 35, pp. 257 and 277. It may be that Newton, as a mathematician, wanted to add quantities which had the same *dimensionality*. However, by the time the *Principia* was completed the first law of motion was stated correctly, as it had been proposed earlier by Descartes. Newton coined the name *centripetal force* after his correspondence with Hooke, who referred to a force as *impressed* power. Previously, Newton had been guided by the concept of a *centrifugal force* (see Ref. 26), introduced also independently by Huygens, Today this is regarded as an *effective force* in a rotating frame of reference. Huygens apparently did not fully understand the concept of gravitational attraction as an external impressed force, and for this reason he may have missed discovering the inverse square law dependence of the gravitational force from Kepler's third law.

[66]R. Hooke, Ref. 23, p. 185.

[67]Recently a controversy was initiated by R. Weinstock, "Dismantling a centuries-old myth: Newton's principia and inverse-square orbits," Am. J. Phys. **50**, 610–617 (1982), who claims that Newton did not prove the inverse problem of dynamics; for references to the literature and a careful study of this question, see B. H. Pourciau, "On Newton's Proof that inverse-square orbits must be conics," Ann. Sci., **48**, 159–172 (1991); "Newton's solution of the one body problem," Arch. Hist. Exact Sci. **44**, 125–146 (1992); M. Nauenberg, "The mathematical principles underlying Newton's Principia revisited" (to be published).

[68]The vertices for the polygonal construction on the circle are not labeled in Hooke's diagram, Fig. 1, but are referred to in the accompanying text by the latin letters *a*, *b*, *k*. Hooke is not entirely consistent in his notation, because he also uses the letters *a* and *h* for two of the vertices associated with the elliptic trajectory.

[69]In the corresponding fundamental construction in the *Principia*, Book I, Section II, Proposition I, Theorem I, Newton likewise leaves out an explicit statement of the proportionality of the change in momentum to the magnitude of the time interval. However this is included later in Proposition VI, theorem V, where the limit of small time steps is discussed, corresponding to a continuous trajectory and force.

[70]A. R. Hall, "Newton on the calculation of central forces," Ann. Sci. **13**, 62–71 (1957).

[71]C. Huygens, *De Vi Centrifuga*, in Ouvres Complètes de Christiaan Huygens XVI, (The Hague, 1929), pp. 253–301.

[72]This error in the graphical location of the foci is revealing as it gives some evidence that Hooke may not have fully appreciated the affine transformation by which the ellipse and the associated polygonal path can be obtained from the circle. An affine transformation factor of 1/2 would have given him the exact location of the foci (at a distance from the center O by a factor $\surd(3)/2$ of the major semiaxis of the ellipse).

[73]According to the biography by Richard Waller (Ref. 22), written two years after Hooke had died, he tells us that as a young boy "···he fell seriously upon the study of the mathematicks, the Dr. (Busby) encouraging him therein, and allowing him particular times for that purpose. In this he took the most regular Method, and first made himself *Master of Euclide's* Elements, and thence proceeded orderly from that sure Basis to the other parts of the Mathematicks, and after to the application thereof to Mechanicks, his first and last Mistress···" From his friend Aubrey we learn that "···in one week's time made himselfe master of the first VI bookes of Euclid···" Later on Hooke became Professor of Geometry at Gresham College in London.

[74]An important property of the discrete Newton-Hooke algorithm is that it corresponds to a time dependent *canonical or simplectic* transformation. Therefore, even for finite δt, it conserves a fundamental property of Hamiltonian dynamics. See, for example, V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, Berlin, 1984). This property accounts for the improved convergence of this low order algorithm over Euler's approximation.

[75]*The Mathematical Papers of Isaac Newton 1684–1691*, edited by D. T. Whiteside (Cambridge University, Cambridge, 1974), Vol. 6, p. 348; Whiteside points out that the evaluation of the integral for the gravitational force is contained among Newton's results in his 1671 tract 'Catalogus posterior, Ordo octavus, problem 9, in *The Mathematical Papers of Isaac Newton*, edited by D. T. Whiteside (Cambridge University, Cambridge), p. 252.

[76]Memoire de l'Academie Royale des Sciences 1710, pp. 519–533. This calculation led to some controversy. The English mathematician John Keill wrote to Newton in 1713 that (Ref. 12)

"Since I left London I have considered Mr. Bernoulli's solution of the Inverse Probleme about Centripetal Forces, and I am amazed at his impudence···He gives a formula for the element of the angle at the center which seemed to be more intricate than yours, but I find it to be only yours disguised, so that his general solution is only taken from yours, and he has done nothing but what you had done better before. In his application of it to the particular case (of an inverse-square force) he has with a great deal of Labour showed that the curve described must be a *Conick Section* when the thing may be demonstrated in a few lines···"

In the *Principia* Newton did not show explicitly that for an inverse square force, the general integral for central forces, given in Proposition XLI, Problem XXVIII corresponds to *a Conick Section.*

[77]The term *indivisibles* was used by Bonaventura Cavaliere in the title of his book *Geometria Indivisibilibus Continuorum* published in 1635, and later by J. Wallis in his *Arithmetica Infinitorum* (Oxon, 1656) which is devoted to the integration of curves by *the method of indivisibles.* Newton's early mathematical studies were also influenced by the work of Wallis, see *The Mathematical Work of John Wallis* by J. F. Scott (Chelsea Publishing, New York).

[78]H. W. Robinson and W. A. Adams, *The Diary of Robert Hooke 1672–1680* (Taylor and Francis, London, 1935).

[79]The handwriting is that of C. Hayes, who translated the first text on differential calculus, written in French by l'Hospital (Ref. 80) under the title *A treatise on fluxions* (D. T. Whiteside, private communication).

[80]Guillaume François Marquis de l'Hospital, et du Montellier, Comte de Saintemesme, et d'Antremontes, Signeur d'Ougues, et autres liex, *Analyse des Infiniment Petits*, Pour l'intelligence des lignes courbes, (A Paris,

de l'Imprimerie Royale, MDCXCVI). It was also translated and published in 1730 by E. Stone under the title *The Method of Fluxions both Direct and Inverse* (The Former being a translation from···, and the later supply'd by the Translator). Newton's original work was first published by John Colson in English in 1736, in a book entitled *The Method of Fluxions and Infinite Series* with its Application to the Geometry of Curve-Lines, By the Inventor Sir Isaac Newton, Kt., Late president of the Royal Society. Translated from the author's Latin Original not yet made publick. London, Printed by Henry Woodfall; M.DCC.XXXVI.

[81]D. J. Struik, "The origin of l'Hospital's rule," Math. Teacher **56**, 257–260 (1963).

[82]G. W. Leibniz, *Nova Methodus pro Maximis et minimis*···Acta Eruditorum, 467-473 (Anno MDCLXXXIV).

[83]Hooke's biographer Waller informs us that after about 1681/2 "···From this time, or rather something before, he began to be more reserv'd than he had been formerly, so that altho' he often made Experiments, and shew'd new Instruments and Inventions, and read his *Cutlerian* Lectures, yet he seldom left any full Account of them to be enter'd designing, as he said, to fit them himself for the Press, and then make them publick, which he never perform'd. This is the reason that I am oblig'd to be the shorter in the remainding part of his Life; and shall only touch upon some few of his Performances, since the bare nameing of them, or mentioning their Titles, will but create an uneasy Curiosity in the Reader without any satisfaction···In the beginning of the Year 1687, his Brothers Daughter, Mrs. Grace Hooke dy'd, who had liv'd with him several Years, the concern for whose Death he hardly ever wore off, being observ'd from that time to grow less active, more Melancholly and Cynical···" Ref. 22, p. 24. Grace was his mistress for some time, and her death followed shortly after the publication of the *Principia*, printed on July, 1686.

[84]At the time these words were written, I. Newton was president of the Royal Society, a post he accepted only after Hooke died, and Richard Waller was the secretary of the Society.

[85]I am indebted to D. W. Dewhirst for help in deciphering some words in Hooke's text.

Iodine molecular constants from absorption and laser fluorescence

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(Received 1 October 1992; accepted 29 October 1993)

Absorption and laser-induced fluorescence spectra of the iodine molecule are compared as sources of molecular constants. The comparative simplicity of the fluorescence spectrum and the increased precision provide a vivid and direct understanding of the iodine molecular ground state.

I. INTRODUCTION

The iodine molecular spectrum was one of the first to be analyzed successfully and provides an ideal case for demonstrating the basic characteristics of diatomic spectra.[1–3] It possesses a long absorption series in the visible region associated with the B–X states which correlate at large separations with the well-known atomic fine structure states, as shown in Fig. 1. This spectrum has remained important in the development of new techniques in molecular spectroscopy following the advent of lasers. Methods of optical pumping in molecules and the development of saturation and polarization spectroscopy were pioneered in iodine and many of its molecular constants are known with considerable precision.[4–8]

The most striking element of the iodine absorption spectrum is the long series of the B–X bands in the visible. The electronic potential energy curves which give rise to the electronic–vibrational–rotational levels involved in the bands are shown as full lines in Fig. 1. The B level is the first member of these electronically excited levels. Levels other than the ground X level which correlate to the ground atomic configuration at large separation are shown as broken lines, although the repulsive states are not yet well established. Both B and X states have $\Omega=0$, (i.e., zero electronic angular momentum along the molecular axis) in standard molecular notation as specified in Chap. IV of Ref. 2. The visible system of B–X bands can be studied in straightforward absorption spectroscopy and yields a great deal of information even when studied with an instrument of modest resolving power.

Absorption experiments on the B–X system have been carried out in student laboratories for many years (see, e.g., Ref. 9) and, while we summarize the information that can be obtained by conventional spectroscopy, the aim of this paper is to describe a laser fluorescence experiment which gives a greatly simplified spectrum. Due to fortuitous coincidences with molecular absorptions, fluorescence from iodine can be stimulated by light from an inexpensive He–Ne laser. The fluorescence spectrum is simpler to analyze than the absorption spectrum and gives immediate evidence for the J-selection rule, anharmonicity of the lower state, and the difference between the bond lengths of the B and X states.

We briefly summarize the analysis of the visible absorp-

